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Liquidity Risk, Speculative Trade, and the Optimal Latency of Financial Markets*

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Garbade and Silber (1979) demonstrate that an asset will be liquid if it has (1) low price volatility and (2) a large number of public investors who trade it. Although these results match nicely with common notions of liquidity, one key element is missing: liquidity also depends on (3) an asset's correlation with other securities. For example, if an illiquid asset is highly correlated with a liquid asset, then speculators will naturally step in and "make it liquid". In this paper, we update Garbade and Silber's model to include an infinitely liquid market security. We show that when the market security is added, the liquidity of the original asset is an increasing function of its correlation with the market. Furthermore, we show that at a critical correlation value of $\sqrt{3/4}$, it is optimal for the asset to continuously clear, i.e., for orders to transact immediately when placed in the market. This low-latency result holds regardless of the other properties of the asset. The updated model can help answer several questions relevant to current financial markets: "How and why do short-term speculators provide liquidity in markets?", "How much benefit do these speculators add?", and "Can extremely low-latency in markets be beneficial?"

Keywords: call auctions; clearing frequency; co-location; high-frequency trading; latency; liquidity.

JEL Classification: G14, G19.

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I Introduction

An asset is commonly considered liquid if it can be traded by an investor in short order at a price close to its equilibrium value. As Garbade and Silber (1979) point out, there are two fundamental factors that can disrupt the trading prices of investors, and therefore, that affect liquidity. The first factor is price volatility. All other things equal, an investor will find it more difficult to trade an asset at an agreeable price if it has high volatility. Therefore, we should expect liquidity to be a decreasing function of volatility. The second factor is market size, i.e., the number of public investors who trade the asset. All other things equal, an investor will find it difficult to trade an asset at an agreeable price if there are few counterparties to trade with. Liquidity, therefore, is affected by market participation, and we should expect liquidity to increase with market size.

There is, however, a third fundamental factor that affects liquidity: the correlation of an asset's value to that of other assets.[1] Indeed, when an otherwise illiquid asset is correlated with a liquid asset, speculators will naturally step in and make the illiquid asset liquid. The process itself is not zero-sum – speculators do not take liquidity from one asset and shift it to the other. In fact, liquidity is enhanced for both assets.

To analyze this process of cross-asset liquidity enhancement, we update the model of Garbade and Silber (1979). We add an infinitely liquid "market security" and study its influence on the liquidity of the non-market asset. As we show below, the liquidity of the non-market asset increases in all cases except when the asset's value is completely uncorrelated with the market. Furthermore, we show that when the correlation exceeds a critical threshold of $\rho = \pm\sqrt{3/4} \approx \pm 0.87$, that it is optimal (from an investor's perspective) for the asset to trade continuously, i.e., with zero latency.

[1] Asymmetric information is another factor that can affect liquidity. Most models produce a negative relationship between asymmetric information and liquidity (e.g., Glosten and Milgrom (1985) and Kyle (1985)). The dynamics are complicated, however, and in fact, liquidity can also increase with asymmetric information (see Vayanos and Wang (2012)). Finally, it should be noted that all of these factors, although more fundamental than variables such as the bid-ask spread or market depth, should, in part, be determined endogenously.

The framework that Garbade and Silber develop is extremely useful for studying liquidity – it abstracts from microstructure variables such as bid-ask spreads and order book volumes and grounds liquidity in more primitive economic variables: the volatility of assets and the aggregate activity of investors within these assets. Furthermore, it sets the target price of an investor to the equilibrium price that held when the investor first decided to trade rather than to a future end-of-period price, which is commonly assumed in other microstructure models (e.g., Glosten and Milgrom (1985) and Kyle (1985)). Direct adverse selection, therefore, is not of concern to the investors. Instead, it is important to transact at a price that minimizes what Garbade and Silber call *liquidity risk* – the variance of the difference between the equilibrium value of an asset at the time a market participant decides to trade and the transaction price the investor ultimately realizes. Such a definition captures well the concerns of many large investors in the market and is the metric we adopt in this paper to measure liquidity.

There are further attributes of Garbade and Silber's model that make it especially relevant to current financial markets. Liquidity provision in their model is a competitive enterprise based on signal extraction from orderflow, and it naturally arises out of speculative activity. More important, their model does not fall apart when liquidity providers are removed from the market (in contrast to many other models in market microstructure). This feature allows direct analysis of the benefits of speculative trade. Such analysis is important for current financial markets, where designated dealers and market makers serve a secondary role to proprietary, automated, high-frequency, and low-latency speculators (see Gerig and Michayluk (2013)). Although there is evidence that some of these *high-frequency traders*, as they are called, do not provide liquidity, the evidence suggests that a majority do, and they do so by conditioning on price movements in correlated securities just as in our model (see Gerig and Michayluk (2013) and references therein).

Finally, Garbade and Silber's framework can be used to study the optimal latency of financial markets. In their model, liquidity risk is directly affected by the interval between market clearings, i.e., latency. Because latency is not a fundamental economic variable, Garbade and Silber treat it as a control variable and determine the optimal latency of the market from an investor's perspective. Although many others have described the low-latency environment in current financial markets as an "arms race" (e.g., Haldane (2011), Farmer and Skouras (2012), and Budish et al. (2013)), our model is the first to demonstrate the benefits of low-latency trade.

The rest of the paper is organized as follows. Section II reviews the relevant literature, Section III presents the baseline model, Section IV analyzes the model with a competitive, risk neutral liquidity provider, Section V analyzes the model with liquidity providers and an additional market security, and Section VI concludes.

II Literature Review

Our paper is particularly related to three research strands: (1) the impact of technological innovations on market quality and the optimal structure of market clearing, (2) the private and social benefits of liquidity provision, and (3) the relationship between the liquidity of an asset and its correlation with the overall market. Below we provide a brief overview of the literature on these three topics.

Our paper is related to the literature focusing on the impact of technological innovations on market quality (e.g., Garbade and Silber (1978) and Easley et al. (2013))[2] and the optimal structure of markets (e.g., Garbade and Silber (1979), Amihud et al. (1997), Kalay et al. (2002)). Garbade and Silber (1978) examine the effects of the introduction of two major innovations in the information transmission of financial markets, namely the establishment of the telegraph (from 1840 onwards) and the establishment of the consolidated tape at the NYSE (in 1975). With the telegraph inter-market price differentials quickly narrowed, whereas the introduction of the consolidated tape did not have a discernible effect.[3] In a similar fashion, Easley

[2] An excellent overview of empirical studies on financial innovation can be found in Frame and White (2004).

[3] The authors speculate that the consolidated tape added little value due to rather efficient pre-existing telecommunication links.

et al. (2013) investigate the effects of an upgrade of the posts on NYSE in 1980. This particular technological change provided off-floor traders with lower latency across different dimensions, i.e., faster order submission and more recent information on trades and quotes. The authors show that this change had significant positive impacts on liquidity, turnover, and returns. The main explanation is that slower off-floor traders could reduce their exposure to adverse selection by conditioning their activity on more recent information.

Amihud et al. (1997) and Kalay et al. (2002) investigate a major change in the trading mechanism of stocks on the Tel Aviv Stock Exchange (TASE). In 1987, trading of large cap stocks on TASE was moved from once-a-day call auctions to an opening call auction followed by iterated continuous trading. As explained in Garbade and Silber (1979), an increase in the clearing frequency has two counteracting effects on liquidity risk: while it allows investors to act on more timely information, trading volume is inter-temporally fragmented. Both studies conclude that the former effect exceeded the latter: market quality, liquidity, and trading volumes increased for large stocks. In contrast, smaller stocks that still traded by call auctions experienced a significant loss in volume relative to the overall market volume. Kalay et al. (2002) conclude that investors prefer continuous to periodic trading, i.e., there is a demand for immediacy. In contrast, Hendershott and Moulton (2011) study a more recent change in the introduction of the NYSE Hybrid Market, which increased automation and significantly reduced the execution time for market orders (from 10 seconds to less than one second). While bid-ask spreads increased, prices became more efficient. In this way, technological change did not increase market quality among all dimensions.

The theoretical literature on the optimal clearing frequency of markets is relatively sparse. To the best of our knowledge, Garbade and Silber (1979) were the first to show that the liquidity risk of the average investor is minimized for intermediate clearing frequencies, i.e., most markets should neither operate continuously nor be cleared very infrequently. Later on, most studies were less concerned with determining the optimal speed of markets, but rather compared continuous and periodic market clearings in general. For example, Madhavan (1992) investigates the performance of order- and quote-driven systems in the different clearing scenarios. Their main finding is that a quote-driven system provides greater price efficiency than a continuous auction system, highlighting the importance of dealers in quote-driven markets. However, with free entry into market making, the equilibria of the two mechanisms coincide. Moreover, the periodic trading mechanism can function when a continuous market fails. More recently, Farmer and Skouras (2012) and Budish et al. (2013) proposed periodic market clearings as a market design response to the high-frequency trading arms race.

Our paper is also related to the literature focusing on the nature and effects of liquidity provision, e.g., Stoll (1978), Ho and Stoll (1980,1981), Pithyachariyakul (1986), and Grossman and Miller (1988). While showing that liquidity provision, i.e. market-making, improves market quality in many cases, the nature of the underlying models used in these papers makes it difficult to directly quantify the effects of market making. In this paper, we can quantify the value of a liquidity provider. Contrary to the standard literature, the liquidity provider in our model has no designated role in the market apart from being able to observe the order flow, and the market still clears without her presence. Therefore, any increase in liquidity when the liquidity provider is present is directly attributable to her.

In this way, our study is related to recent discussions about the effects of high-frequency traders in markets. Many empirical studies indicate that a variety of measures of market quality have improved with the arrival of high-frequency trading (e.g. Hendershott et al. (2011) and Riordan and Storkenmaier (2012)), and this is likely due to their liquidity providing activities as modeled here (e.g., Jovanovic and Menkveld (2012), Menkveld (2013), and Hagströmer and Nordén (2013)).

In our model, we deliberately ignore issues of adverse selection and differentials in speed between investors. Other theory papers have focused on the effects of differential access to speed and how this can increase adverse selection in markets (e.g., Budish et al. (2013) and Bias et al. (2013)). However, the empirical evidence suggests that low-latency trading has exactly the opposite affect on adverse selection and even more so when investors have differential access to speed (see Hasbrouck and Saar (2013) and Brogaard et al. (2013)). Low-latencies in markets, therefore, seem to aide the kind of speculative activity that provides liquidity (i.e., the kind we model here and that is directly modeled as liquidity provision in Gerig and Michayluk (2013)) rather than the bad kind of activity that inhibits the work of liquidity providers.

Lastly, our paper highlights the relationship between the liquidity of an asset and its correlation with the overall market. In a recent empirical study, Chan et al. (2013) show that the liquidity of a security increases with the fraction of volatility due to systematic risk, exactly as predicted in our model. Furthermore, they find that improvement in liquidity following the addition of a stock to the S&P 500 Index is directly related to the stocks increase in correlation with the market.

The only other paper we are aware of that directly models the relationship between the liquidity of an asset and its correlation to other securities is Baruch and Saar (2009). In their model, as in our model, the liquidity provider can form a better estimate of prices when observing order flow from correlated assets. However, they use a multi-asset Kyle (1985) framework and their results are due to reductions in adverse selection costs for the liquidity provider. Because our liquidity provider is a speculator, liquidity enhancement is due to the profit motives of the speculator, i.e., using signals in one security to trade in another, rather than lower adverse selection costs (which we believe to be a more accurate description for modern financial markets).

III Baseline Model

As in Garbade and Silber (1979), we consider a single security that is traded by public investors in a market with periodic clearings. (In later sections, we consider the addition of liquidity providers and also a second security.) The time interval between clearings is τ, and ultimately, we will be interested in determining the optimal τ from an investor's perspective.[4]

Between clearings, investors (indexed in each interval by i) arrive at a constant rate ω and submit excess demand schedules to the market. These demand schedules are unobservable to other investors and remain in the market until the next market clearing. At each clearing, the transaction price is set to the value that clears the market, i.e., to the value that produces zero aggregate excess demand. The excess demand schedule of the ith investor is a linearly increasing function of the reservation price of the investor, r_i, and a linear decreasing function of the clearing price, p,

$$D(p) = a(r_i - p), \tag{1}$$

where a is a positive constant assumed the same for all investors. Note that the ith investor will be a net seller of the security if $r_i < p$ and will be a net buyer if $r_i > p$.

Between any two clearings, a total number $K = \omega\tau$ investors will submit excess demand schedules to the market. The market clearing price is the unique price that sets aggregate excess demand to zero,

$$0 = \sum_{i=1}^{K} a(r_i - p). \tag{2}$$

Rearranging the equation reveals that the clearing price is the average reservation price of the arriving investors,

$$p = \sum_{i=1}^{K} r_i/K. \tag{3}$$

We assume there exists an unobservable equilibrium price for the security, m_t, at all times and that the reservation price of investor i is normally distributed around the prevailing equilibrium price, $m_{t-1+i/\tau}$ (which we denote m_i for short), at the instant

[4]Note that we attempt to keep our notation as consistent as possible with Garbade and Silber's original paper.

the investor decides to trade,[5]

$$
\begin{aligned}
r_i &= m_i + g_i, && (4)\\
g_i &\sim N(0, \sigma^2), && (5)
\end{aligned}
$$

where g_i is assumed to be uncorrelated across investors. We denote by $\bar{r}_t$ the average reservation price of the investors at market clearing t (which is the market clearing price when the market does not contain liquidity providers),

$$
\bar{r}_t = \sum_{i=1}^{K} (m_i + g_i)/K. \tag{6}
$$

We denote by $\bar{m}_t$ the average equilibrium price over the interval, $\bar{m}_t = \sum_i m_i/K$, and we denote by f_t the average of g_i, i.e., $f_t = \sum_i g_i/K$. Note that,

$$
\begin{aligned}
\bar{r}_t &= \bar{m}_t + f_t, && (7)\\
f_t &\sim N(0, \sigma^2/(\omega\tau)). && (8)
\end{aligned}
$$

We assume that the instantaneous equilibrium price m_t evolves as a driftless Brownian motion with variance $(3/2)\psi^2$, i.e., $m_t = (3/2)\psi^2 B_t$ (the prefactor 3/2 is used for convenience and its purpose will become apparent in the following equation). Therefore, the average equilibrium price for investors at clearing t evolves according to the following equation,

$$
\begin{aligned}
\bar{m}_t &= \bar{m}_{t-1} + e_t, && (9)\\
e_t &\sim N(0, \tau\psi^2), && (10)
\end{aligned}
$$

[5] In Garbade and Silber (1979), the investor decides to trade at time $t-1/2$ but has a reservation price that is normally distributed around the future equilibrium price at time t. We have chosen a different setup (which we believe is more natural) where the reservation price of an investor is normally distributed around the instantaneous equilibrium price at the time he/she decides to trade. This departure means that much of our analysis will be based on average equilibrium prices over the interval τ rather than on instantaneous equilibrium prices as in Garbade and Silber (1979).

where we have used the result that the variance of the difference between two consecutive averaged points (each over an interval τ) of a standard Brownian motion is,

$$\mathrm{Var}\left[(1/\tau)\int_{\tau}^{2\tau} B_t\,dt - (1/\tau)\int_{0}^{\tau} B_t\,dt\right] = (2/3)\tau. \tag{11}$$

We assume that e_t is serially uncorrelated and also uncorrelated with g_i and therefore f_t.

A Liquidity Risk

As in Garbade and Silber (1979), we define liquidity risk as the variance of the difference between the equilibrium value of the security when an investor arrives at the market, m_i, and the transaction price ultimately realized for the investor's trade, in this case $\bar{r}_t$.[6] The liquidity risk for investor i in a market without liquidity providers is therefore,

$$\begin{aligned} V_P &= \mathrm{Var}[(\bar{r}_t - \bar{m}_t) + (\bar{m}_t - m_i)], && (12)\\ &= \mathrm{Var}[\bar{r}_t - \bar{m}_t] + \mathrm{Var}[\bar{m}_t - m_i], && (13) \end{aligned}$$

where the two expressions in parentheses separate because there is no covariance between them. The variance of the first term, $\mathrm{Var}[(\bar{r}_t - \bar{m}_t)]$, is just the variance of f_t. For the second term, the variance depends on the arrival time of the investor. If the investor arrives at a point in time that is a fraction ϕ of the total interval τ from the previous clearing, then the variance of the second term will be,

$$\begin{aligned} \mathrm{Var}\left[\bar{m}_t - m_i\right] &= \mathrm{Var}\left[\left(\int_0^{\phi\tau} (3/2)\psi^2 B_t\,dt + \int_0^{\tau-\phi\tau} (3/2)\psi^2 B_t\,dt\right)\Big/\tau\right], && (14)\\ &= (1/2)\left[\phi^3 + (1-\phi)^3\right]\tau\psi^2. && (15) \end{aligned}$$

[6] Grossman and Miller (1988) use a very similar definition of liquidity risk.

If the investor arrives at the beginning or end of the interval ($\phi = 0$ or $\phi = 1$), then the variance is at its maximum value, $(1/2)\tau\psi^2$, and if the investor arrives in the middle of the interval ($\phi = 1/2$), the variance is at its minimum value, $(1/8)\tau\psi^2$. The final equation for liquidity risk in a market of public investors is therefore,

$$
\begin{aligned}
V_P &= \mathrm{Var}[(\bar{r}_t - \bar{m}_t) + (\bar{m}_t - m_i)], && (16)\\
&= \mathrm{Var}[\bar{r}_t - \bar{m}_t] + \mathrm{Var}[\bar{m}_t - m_i], && (17)\\
&= \sigma^2/(\omega\tau) + (1/2)\ \ \phi^3 + (1-\phi)^3\ \ \tau\psi^2. && (18)
\end{aligned}
$$

If we assume that the timing of an investor's trading decision is uncorrelated with the timing of market clearings, we can average over all ϕ in the interval $[0, 1]$, which gives $\int_0^1 (\phi^3 + (1-\phi)^3) = 1/2$. Liquidity risk is therefore,

$$V_P = \sigma^2/(\omega\tau) + \tau\psi^2/4. \tag{19}$$

Because our setup is different than Garbade and Silber (1979), our equation for liquidity risk is slightly different (specifically, the denominator of the second term in their paper is 2 instead of 4). Notice that liquidity risk is increasing in the volatility of the security, increasing in the variance of investor reservation prices, and decreasing in the frequency of investor arrival. The effect of the clearing frequency ($1/\tau$) on liquidity risk is nonlinear. When market clearings are frequent, this decreases the difference between the clearing price and the average equilibrium price of the security, but it also increases the difference between the average equilibrium price of the security and the specific equilibrium price used as a reference by the investor. There is a "Goldilocks" value for τ that optimizes the tradeoff between these two effects, and we determine this value below.

The optimal trading interval τ_P^* from an investor's perspective is just the value of τ that minimizes liquidity risk. This value can be found by taking the derivative of

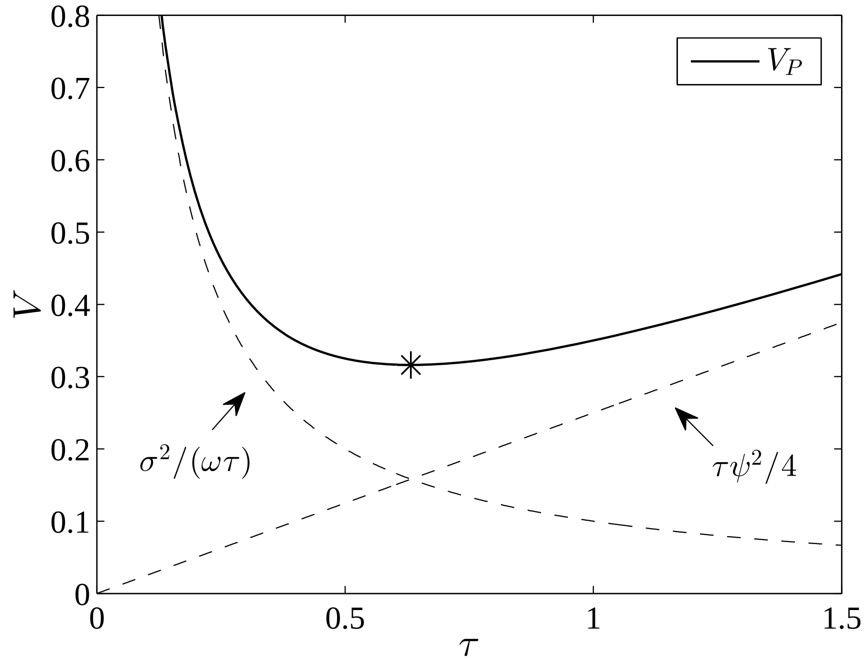

Figure 1: Liquidity risk, V_P, as a function of the time between market clearings, τ, in a public market without a liquidity provider. Parameters used in the plot are $\psi = 1$, $\sigma = 1$, and $\omega = 10$. The optimal point (V_P^*, τ_P^*) is shown with an asterisk. Also shown are the components of liquidity risk $\sigma^2/(\omega\tau)$ and $\tau\psi^2/4$.

liquidity risk with respect to τ and setting to zero,

$$\tau_P^* = 2\,\frac{\sigma/\omega^{1/2}}{\psi}. \tag{20}$$

The minimum value of liquidity risk, $V_P* = V_P(\tau_P^*)$, is,

$$V_P^* = \left(\sigma/\omega^{1/2}\right)\psi. \tag{21}$$

In Fig. 1, we show liquidity risk as a function of the time between market clearings, τ, when $\psi = 1$, $\sigma = 1$, and $\omega = 10$. We also show the optimal point (V_P^*, τ_P^*).

IV Model with a Liquidity Provider

As discussed in Garbade and Silber (1979), enterprising individuals (i.e., speculators) can devise a better estimate for the equilibrium price than is contained in the market clearing price r_t and can profit by buying and selling according to this estimate. In effect, these speculators act as liquidity providers in the market.

Here, we assume that a single competitive and risk neutral liquidity provider (or speculator) exists, that she observes the aggregate excess demand of the market directly before the market is cleared, and that she submits an excess demand schedule at each market clearing such that the clearing price always equals her estimate of the equilibrium price. Many of the seminal market microstructure papers published after Garbade and Silber (1979) (such as Kyle (1985) and Glosten and Milgrom (1985)) assume the same type of competitive, risk neutral liquidity provider. However, in these other models, the benefit of the liquidity provider cannot be analyzed, whereas it can in Garbade and Silber's framework. Below we show that the liquidity provider reduces the minimum liquidity risk of public investors by a factor of 1.5. In the next section, we show that when the liquidity provider is further enabled so that she observes the price of the "market", liquidity risk is reduced even further.

A Liquidity Risk

The liquidity provider will form an estimate of the average equilibrium price over the interval, which we denote by $\hat{m}_t$, and will submit a demand schedule that forces the clearing price to this value. Therefore, in the equation for liquidity risk, the clearing price is $\hat{m}_t$ instead of $\bar{r}_t$.

The model with a liquidity provider is a special case of the model presented in the next section. Here, we just present results for liquidity risk and leave details of the derivation to the next section and the Appendix.

$$\begin{aligned}
V_L &= \mathrm{Var}[(\hat{m}_t - \bar{m}_t) + (\bar{m}_t - m_i)], && (22)\\
&= \mathrm{Var}[\hat{m}_t - \bar{m}_t] + \mathrm{Var}[\bar{m}_t - m_i] + 2\ \mathrm{Cov}[\hat{m}_t - \bar{m}_t, \bar{m}_t - m_i], && (23)\\
&= \frac{2\,(\phi_1 + 2\phi_2)\,\tau\psi^2 + 2(\phi_1 - 2\phi_2)\tau\psi^2\sqrt{1 + \frac{4\sigma^2/\omega}{\tau^2\psi^2}} + 4\sigma^2/(\omega\tau)}{2\left(1 + \sqrt{1 + \frac{4\sigma^2/\omega}{\tau^2\psi^2}}\right)}, && (24)
\end{aligned}$$

where,

$$\begin{aligned}
\phi_1 &\equiv (1/2)\ \ \phi^3 + (1-\phi)^3\ \ , && (25)\\
\phi_2 &\equiv (1/4)\ \ \phi^3 + 2(1-\phi)^3 + 3(1-\phi)\phi^2\ \ . && (26)
\end{aligned}$$

If the investor's arrival time is not correlated with the timing of market clearings, then liquidity risk is the expectation over ϕ,

$$V_L \;=\; \frac{(1/2+)\,\tau\psi^2 + (1/2)\tau\psi^2\sqrt{1 + \frac{4\sigma^2/\omega}{\tau^2\psi^2}} + 4\sigma^2/(\omega\tau)}{2\left(1 + \sqrt{1 + \frac{4\sigma^2/\omega}{\tau^2\psi^2}}\right)}. \tag{27}$$

A plot of $V_L(\tau)$ is shown later in Fig. 3. The optimal trading interval τ_L^* is,

$$\tau_L^* = \quad \frac{2}{\sqrt{3}} \quad \frac{\sigma/\omega^{1/2}}{\psi}, \tag{28}$$

and the minimum value of liquidity risk is,

$$V_L^* = \quad \frac{7}{6\sqrt{3}} \quad \sigma/\omega^{1/2} \quad \psi. \tag{29}$$

Notice that with the liquidity provider, the optimal clearing frequency $(1/\tau_L^*)$ increases by a factor of $\sqrt{3} \approx 1.7$ from the public market case (regardless of the other parameters). In addition, the liquidity provider reduces liquidity risk by a factor of $6\sqrt{3}/7 \approx 1.5$, again regardless of the values of other parameters in the model.

V Model with a Liquidity Provider and Market Information

In general, for a market of N securities, the average reservation price of the different securities at market clearing t can be written,

$$\begin{aligned} \bar{\mathbf{r}}_t &= \bar{\mathbf{m}}_t + \mathbf{f}_t, &(30)\\ \mathbf{f}_t &\sim N(0, \mathbf{\Sigma}), &(31) \end{aligned}$$

and the average equilibrium price over the market clearing interval can be written,

$$\begin{aligned} \bar{\mathbf{m}}_t &= \bar{\mathbf{m}}_{t-1} + \mathbf{e}_t, &(32)\\ \mathbf{e}_t &\sim N(0, \mathbf{\Psi}), &(33) \end{aligned}$$

where $\bar{\mathbf{r}}$, $\bar{\mathbf{m}}$, $\bar{\mathbf{f}}$, and $\bar{\mathbf{e}}$ are $N \times 1$ vectors and $\mathbf{\Sigma}$ and $\mathbf{\Psi}$ are $N \times N$ matrices.

For a market of relatively few securities, it is not too difficult to calculate estimates of $\bar{\mathbf{m}}_t$ (denoted $\hat{\mathbf{m}}_t$) and to determine liquidity risk when $\mathbf{\Sigma}$ and $\mathbf{\Psi}$ are fully specified. The process involves numerically solving the appropriate discrete time algebraic Riccati equation (see the Appendix) and then using this solution in straightforward equations. Analytic results, however, are often extremely messy – even for just two securities.

In order to present analytic results, we treat the model with a liquidity provider in a market with many assets as a special case of a two security market where the second security is the "market security",

$$\bar{\mathbf{r}}_t = \begin{pmatrix} \bar{r}_t \\ \bar{r}_{M,t} \end{pmatrix} \qquad \bar{\mathbf{m}}_t = \begin{pmatrix} \bar{m}_t \\ \bar{m}_{M,t} \end{pmatrix} \tag{34}$$

$$\mathbf{f}_t = \begin{pmatrix} f_t \\ f_{M,t} \end{pmatrix} \qquad \mathbf{\Sigma} = \begin{pmatrix} \sigma^2/(\omega\tau) & \varrho\sigma\sigma_M/(\sqrt{\omega\omega_M}\tau) \\ \varrho\sigma\sigma_M/(\sqrt{\omega\omega_M}\tau) & \sigma_M^2/(\omega_M\tau) \end{pmatrix} \tag{35}$$

$$\mathbf{e}_t = \begin{pmatrix} e_t \\ e_{M,t} \end{pmatrix} \qquad \mathbf{\Psi} = \begin{pmatrix} \tau\psi^2 & \rho\tau\psi\psi_M \\ \rho\tau\psi\psi_M & \tau\psi_M^2 \end{pmatrix}, \tag{36}$$

where ϱ is the correlation of the difference between reservation prices and equilibrium prices across the two assets, and ρ is the correlation of equilibrium price changes across the two assets. We make an idealized assumption that order flow for the market security is so frequent that $\omega_M \gg 1$ and,

$$\mathbf{\Sigma} \approx \begin{pmatrix} \sigma^2/(\omega\tau) & 0 \\ 0 & 0 \end{pmatrix} \tag{37}$$

The liquidity provider, therefore, has perfect information about the average equilibrium price of the market security at each clearing.

A Liquidity Risk

Liquidity risk is,

$$\begin{aligned} V_M &= \mathrm{Var}[(\hat{m}_t - \bar{m}_t) + (\bar{m}_t - m_i)], & (38) \\ &= \mathrm{Var}[\hat{m}_t - \bar{m}_t] + \mathrm{Var}[\bar{m}_t - m_i] + 2\ \mathrm{Cov}[\hat{m}_t - \bar{m}_t, \bar{m}_t - m_i], & (39) \\ &= \mathbf{S}_{(1,1)} + \phi_1 \tau \psi^2 + 2(\mathbf{G}_{(1,1)} - 1)\phi_2 \tau \psi^2 + 2\mathbf{G}_{(1,2)} \phi_2 \rho \tau \psi \psi_M, & (40) \end{aligned}$$

where $\mathbf{S}_{(1,1)}$, $\mathbf{G}_{(1,1)}$, and $\mathbf{G}_{(1,2)}$ are the respective elements of the matrices used in the Kalman filter when solving for $\hat{m}_t$. A derivation of this equation is given in the Appendix.

Solving the Riccati equation and plugging into Eq.40 (see the Appendix),

$$V_M = \frac{2\left(\phi_1 + 2\phi_2\Theta\right)\tau\psi^2 + 2(\phi_1 - 2\phi_2\Theta)\tau\psi^2 \sqrt{1 + \frac{4\sigma^2/\omega}{\Theta\tau^2\psi^2}} + 4\sigma^2/(\omega\tau)}{2\left(1 + \sqrt{1 + \frac{4\sigma^2/\omega}{\Theta\tau^2\psi^2}}\right)}, \tag{41}$$

where $\Theta \equiv 1 - \rho^2$. Again, if we assume that the investor's arrival time is not correlated

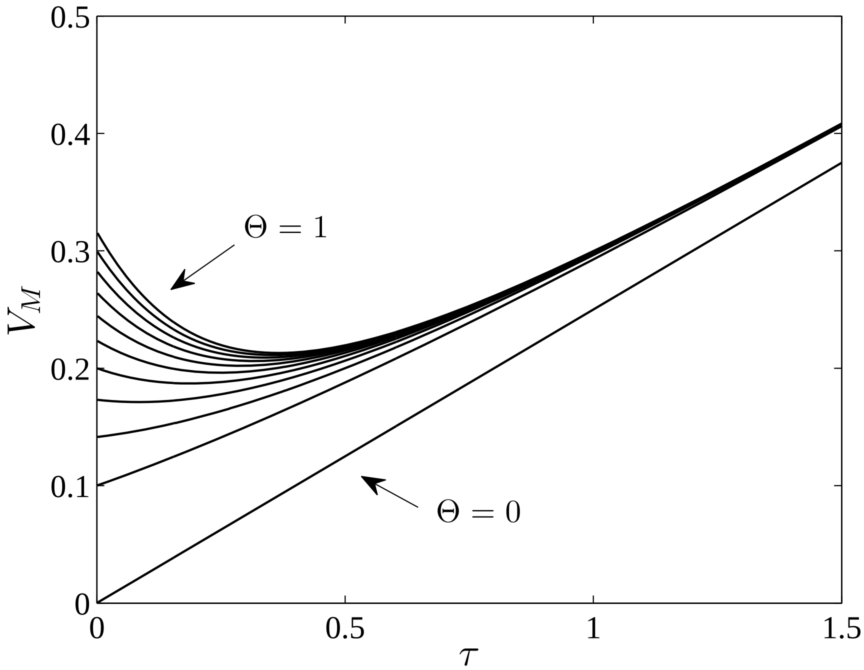

Figure 2: Liquidity risk, V_M, as a function of the time between market clearings, τ, in a market with a liquidity provider and market information. Curves are shown for parameters $\psi = 1$, $\sigma = 1$, $\omega = 10$, and with $\Theta = 0$ to $\Theta = 1$ in increments of 0.1.

with the timing of market clearings, then liquidity risk is the expectation over ϕ,

$$V_M = \frac{(1/2 + \Theta)\,\tau\psi^2 + (1/2 - \Theta)\tau\psi^2\sqrt{1 + \frac{4\sigma^2/\omega}{\Theta\tau^2\psi^2}} + 4\sigma^2/(\omega\tau)}{2\left(1 + \sqrt{1 + \frac{4\sigma^2/\omega}{\Theta\tau^2\psi^2}}\right)} \tag{42}$$

In Fig. 2, we show liquidity risk, V_M, as a function of the time between market clearings, τ, when $\psi = 1$, $\sigma = 1$, $\omega = 10$, and with $\Theta = 0$ to $\Theta = 1$ in increments of 0.1. Liquidity risk decreases as the correlation of the asset with the market increases (i.e., as Θ decreases). When the asset is perfectly correlated with the market, $\Theta = 0$, liquidity risk is simply the line $\tau\psi^2/4$ and there is no risk when markets continuously clear, $\tau = 0$. When the asset is uncorrelated with the market, $\Theta = 1$, liquidity risk is the same as if the market security was absent, $V_M = V_L$.

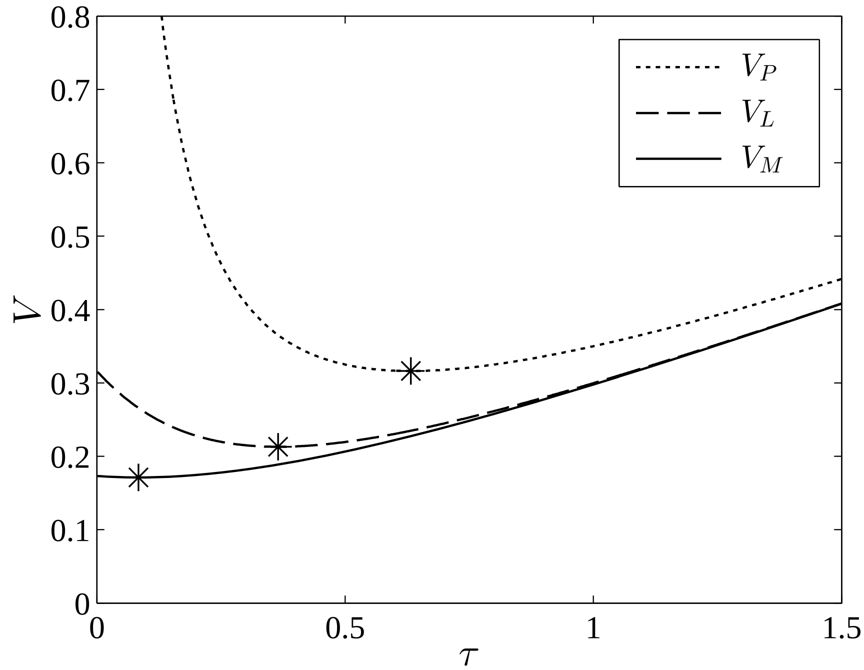

Figure 3: A comparison of liquidity risk, V, for the three models studied in the text. Parameters used in the plot are $\psi = 1$, $\sigma = 1$, $\omega = 10$, and $\Theta = 0.3$. The optimal points (V_P^*, τ_P^*), (V_L^*, τ_L^*), and (V_M^*, τ_M^*) are shown with asterisks.

The optimal trading interval τ_M^* is,

$$\tau_M^* = h_1(\Theta)\, \frac{\sigma/\omega^{1/2}}{\psi}, \tag{43}$$

where,

$$h_1(\Theta) = \frac{\sqrt{1 - 32\Theta + 12\Theta^2 + \sqrt{1 + 20\Theta + 4\Theta^2} + 6\Theta\sqrt{1 + 20\Theta + 4\Theta^2}}}{2\sqrt{3}\Theta}. \tag{44}$$

This equation goes to zero at the critical value $\Theta^c = 1/4$, i.e., when $\rho = \sqrt{3/4} \approx 0.87$. From then on, it is optimal for markets to clear continuously.

For $\Theta > \Theta^c$, the minimum liquidity risk is,

$$V_M^{*+} = h_2(\Theta)/\omega^{1/2} + h_3(\Theta) \;\; \sigma/\omega^{1/2} \;\; \psi, \tag{45}$$

where $h_2(\Theta)$ and $h_3(\Theta)$ are rather complicated functions. For $\Theta \leq \Theta^c$, liquidity risk is minimized when markets continuously clear, i.e., when $\tau = 0$. The equation for liquidity risk when $\Theta \leq \Theta^c$ is,

$$V_M^{*-} = \sqrt{\Theta} \quad \sigma/\omega^{1/2} \quad \psi, \tag{46}$$

In Fig. 3, we compare liquidity risk for the three models studied in the text. Parameters used in the plot are $\psi = 1$, $\sigma = 1$, $\omega = 10$, and $\Theta = 0.3$. We also show the optimal points (V_P^*, τ_P^*), (V_L^*, τ_L^*), and (V_M^*, τ_M^*). Notice how liquidity risk decreases with the addition of the liquidity provider and reduces even further when the market security is added.

VI Conclusions

Although the paper by Garbade and Silber (1979) is more than 30 years old, it provides an excellent framework to study liquidity. Their model is especially relevant for current financial markets, where most liquidity provision occurs through the speculative activity of low-latency/high-frequency traders rather than through the activity of designated liquidity providers. These speculators make trading decision based on estimates of prices and investor order flow across thousands of continuously traded securities, often using a very similar form of the Kalman filter presented here.

As we have demonstrated, when Garbade and Silber's model is updated to include a market security, the liquidity of the non-market asset is improved. Our results, therefore, bring attention to an additional fundamental economic factor that affects the liquidity of assets – the correlation structure of the market. The implications of this relationship for asset pricing and other areas of economics and finance are an interesting unexplored area of future research.

In addition to analyzing this liquidity/correlation relationship, we demonstrate that at a critical threshold value of correlation, it is optimal from an investor's perspective for markets to continuously clear, i.e., for there to be zero latency in markets.

Although many others have described the low-latency environment in current financial markets as an "arms race", our model demonstrates exactly how low-latency trade can be beneficial. A full analysis would involve quantifying this benefit in relation to the cost, which is perhaps an important question to be addressed in future research.

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APPENDIX

The Kalman Filter

The following is a straightforward application of the Kalman filter for the estimation of $\bar{\mathbf{m}}_t$ using contemporaneous and lagged values of $\bar{\mathbf{r}}_t$ (see Meinhold and Singpurwalla (1983)). The *observation equation* is,

$$\begin{aligned} \bar{\mathbf{r}}_t &= \bar{\mathbf{m}}_t + \mathbf{f}_t, && (47) \\ \mathbf{f}_t &\sim N(0, \mathbf{\Sigma}). && (48) \end{aligned}$$

and the *system equation* is,

$$\begin{aligned} \bar{\mathbf{m}}_t &= \bar{\mathbf{m}}_{t-1} + \mathbf{e}_t, && (49) \\ \mathbf{e}_t &\sim N(0, \mathbf{\Psi}). && (50) \end{aligned}$$

Denote by $\hat{\mathbf{m}}_t$ the estimate of $\bar{\mathbf{m}}_t$ based on $\{\bar{\mathbf{r}}_t, \bar{\mathbf{r}}_{t-1}, \bar{\mathbf{r}}_{t-2}, \dots\}$. It can be shown that,

$$\begin{aligned} P(\bar{\mathbf{m}}_t | \bar{\mathbf{r}}_t, \bar{\mathbf{r}}_{t-1}, \dots) &\sim N(\hat{\mathbf{m}}_{t-1} + \mathbf{G}_t[\bar{\mathbf{r}}_t - \hat{\mathbf{m}}_{t-1}],\ \mathbf{S}_t), && (51) \\ P(\bar{\mathbf{m}}_{t+1} | \bar{\mathbf{r}}_t, \bar{\mathbf{r}}_{t-1}, \dots) &\sim N(\hat{\mathbf{m}}_t, \mathbf{R}_{t+1}). && (52) \end{aligned}$$

where $\mathbf{G}_t$ is known as the *Kalman gain* and,

$$\begin{aligned} \mathbf{G}_t &= \mathbf{R}_t(\mathbf{R}_t + \mathbf{\Sigma})^{-1}, && (53) \\ \mathbf{R}_{t+1} &= \mathbf{S}_t + \mathbf{\Psi}, && (54) \\ \mathbf{S}_t &= \mathbf{R}_t - \mathbf{G}_t\mathbf{R}_t. && (55) \end{aligned}$$

The best estimate of $\bar{\mathbf{m}}_t$ based on $\{\bar{\mathbf{r}}_t, \bar{\mathbf{r}}_{t-1}, \bar{\mathbf{r}}_{t-2}, \dots\}$ is just the mean of the distribution $P(\bar{\mathbf{m}}_t | \bar{\mathbf{r}}_t, \bar{\mathbf{r}}_{t-1}, \dots)$,

$$\hat{\mathbf{m}}_t = \hat{\mathbf{m}}_{t-1} + \mathbf{G}_t(\bar{\mathbf{r}}_t - \hat{\mathbf{m}}_{t-1}). \quad (56)$$

The estimation variance is

$$\mathrm{Var}[\hat{\mathbf{m}}_t - \bar{\mathbf{m}}_t] = \mathbf{S}_t. \tag{57}$$

In general, the above equations are solved iteratively, starting at time zero. Here, we search for convergence of the estimation variance to a limiting value, i.e., we search for a solution when $\mathbf{R}_{t+1} = \mathbf{R}_t$. Rearranging the above equations and setting $\mathbf{R} = \mathbf{R}_{t+1} = \mathbf{R}_t$ produces the following equation,

$$\mathbf{R}(\mathbf{R} + \mathbf{\Sigma})^{-1}\mathbf{R} - \mathbf{\Psi} = 0, \tag{58}$$

which is a version of the discrete time algebraic Riccati equation. The conditions required for a solution to exist are discussed in Anderson and Moore (2005). Note that when $\mathbf{R}$ has reached its steady state, that $\mathbf{G}$ and $\mathbf{S}$ will also be steady. Once $\mathbf{R}$ is determined, then $\mathbf{G}$ and $\mathbf{S}$ can be calculated as follows,

$$\mathbf{G} = \mathbf{\Psi}\mathbf{R}^{-1}, \tag{59}$$

$$\mathbf{S} = \mathbf{R} - \mathbf{\Psi}. \tag{60}$$

Solving the Riccati Equation

In the model with a liquidity provider who does not have access to market information, all variables in the Kalman filter are scalars. Furthermore,

$$\Sigma = \sigma^2/(\omega\tau), \tag{61}$$

$$\Psi = \tau\psi^2. \tag{62}$$

The Riccati equation is therefore,

$$R^2/(R + \sigma^2/(\omega\tau)) - \tau\psi^2 = 0, \tag{63}$$

Solving for R and the rest of the variables in the Kalman filter,

$$
\begin{aligned}
R &= (1/2)\left[\tau\psi^2 + \sqrt{\tau^2\psi^4 + 4\psi^2\sigma^2/\omega}\right], && (64) \\
G &= \frac{2\tau\psi^2}{\tau\psi^2 + \sqrt{\tau^2\psi^4 + 4\psi^2\sigma^2/\omega}}, && (65) \\
S &= (1/2)\left[\sqrt{\tau^2\psi^4 + 4\psi^2\sigma^2/\omega} - \tau\psi^2\right], && (66)
\end{aligned}
$$

In the model with a liquidity provider who has access to market information, we have,

$$
\mathbf{\Sigma} = \begin{pmatrix} \sigma^2/(\omega\tau) & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{\Psi} = \begin{pmatrix} \tau\psi^2 & \rho\tau\psi\psi_M \\ \rho\tau\psi\psi_M & \tau\psi_M^2 \end{pmatrix}. \tag{67}
$$

Solving the Riccati equation,

$$
\begin{aligned}
\mathbf{R} &= \begin{pmatrix} (1/2)\left[(2-\Theta)\tau\psi^2 + \Theta\tau\psi^2\sqrt{1+\frac{4\sigma^2/\omega}{\Theta\tau^2\psi^2}}\right] & \rho\tau\psi\psi_M \\ \rho\tau\psi\psi_M & \tau\psi_M^2 \end{pmatrix}, && (68) \\
\mathbf{G} &= \begin{pmatrix} \frac{2}{-1+\sqrt{1+\frac{4\sigma^2/\omega}{\Theta\tau^2\psi^2}}} & \frac{\left(-1+\sqrt{1+\frac{4\sigma^2/\omega}{\Theta\tau^2\psi^2}}\right)\rho\tau\psi\psi_M}{\left(1+\sqrt{1+\frac{4\sigma^2/\omega}{\Theta\tau^2\psi^2}}\right)\tau\psi_M^2} \\ 0 & 1 \end{pmatrix}, && (69) \\
\mathbf{S} &= \begin{pmatrix} (1/2)\left[\Theta\tau\psi^2\left(-1+\sqrt{1+\frac{4\sigma^2/\omega}{\Theta\tau^2\psi^2}}\right)\right] & 0 \\ 0 & 0 \end{pmatrix} && (70)
\end{aligned}
$$

where $\Theta \equiv 1-\rho^2$. Note that when the security is uncorrelated with the market, i.e., $\Theta = 1$, that the elements $\mathbf{R}_{(1,1)}$, $\mathbf{G}_{(1,1)}$, and $\mathbf{S}_{(1,1)}$ all reduce to the values found in the case when the liquidity provider has no market information (Eqs. 64-66).

Liquidity Risk

The equation for the liquidity risk of an investor trading the security when a liquidity provider is present is,

$$
\begin{aligned}
V_{L,M} &= \mathrm{Var}[(\hat{m}_t - \bar{m}_t) + (\bar{m}_t - m_i)], && (71)\\
&= \mathrm{Var}[\hat{m}_t - \bar{m}_t] + \mathrm{Var}[\bar{m}_t - m_i] + 2\ \mathrm{Cov}[\hat{m}_t - \bar{m}_t, \bar{m}_t - m_i]. && (72)
\end{aligned}
$$

We will start with the first term, $\mathrm{Var}[(\hat{m}_t - \bar{m}_t)]$. The estimation variance of $\bar{\mathbf{m}}_t$ is just $\mathbf{S}$ (see Eq.57). For the security, the variance is reported at position $(1, 1)$,

$$
\mathrm{Var}[\hat{m}_t - \bar{m}_t] = \mathbf{S}_{(1,1)}. \tag{73}
$$

The second term is derived in the text (Eq. 15),

$$
\begin{aligned}
\mathrm{Var}\,[\bar{m}_t - m_i] &= (1/2)\ \ \phi^3 + (1-\phi)^3\ \ \tau\psi^2, && (74)\\
&= \phi_1\tau\psi^2. && (75)
\end{aligned}
$$

where $\phi_1 \equiv (1/2)\,[\phi^3 + (1-\phi)^3]$.

The third term, $2\mathrm{Cov}[\hat{m}_t - \bar{m}_t, \bar{m}_t - m_i]$, can be derived as follows. Subtracting $\bar{\mathbf{m}}_t$ from both sides of Eq. 56 and rearranging,

$$
\hat{\mathbf{m}}_t - \bar{\mathbf{m}}_t = (\mathbf{I} - \mathbf{G}_t)\,(\hat{\mathbf{m}}_{t-1} - \bar{\mathbf{m}}_{t-1}) + \mathbf{G}_t\,(\bar{\mathbf{r}}_t - \bar{\mathbf{m}}_t) + (\mathbf{G}_t - \mathbf{I})\,(\bar{\mathbf{m}}_t - \bar{\mathbf{m}}_{t-1})\,, \tag{76}
$$

where $\mathbf{I}$ is the identity matrix. The elements in the vectors $(\mathbf{I} - \mathbf{G}_t)(\hat{\mathbf{m}}_{t-1} - \bar{\mathbf{m}}_{t-1})$ and $\mathbf{G}_t\,(\bar{\mathbf{r}}_t - \bar{\mathbf{m}}_t)$ are uncorrelated with $(\bar{m}_t - m_i)$ so we can disregard them. In the last vector, $(\mathbf{G}_t - \mathbf{I})\,(\bar{\mathbf{m}}_t - \bar{\mathbf{m}}_{t-1})$, the relevant contribution to $\hat{m}_t - \bar{m}_t$ is the first element,

$$
(\mathbf{G}_{(1,1)} - 1)(\bar{m}_t - \bar{m}_{t-1}) + \mathbf{G}_{(2,1)}(\bar{m}_{M,t} - \bar{m}_{M,t-1}). \tag{77}
$$

The covariance of the random terms in this equation with $(\bar{m}_t - m_i)$ are,

$$\begin{aligned} \mathrm{Cov}[\bar{m}_t - \bar{m}_{t-1}, \bar{m}_t - m_i] &= \phi_2 \tau \psi^2, && (78) \\ \mathrm{Cov}[\bar{m}_{M,t} - \bar{m}_{M,t-1}, \bar{m}_t - m_i] &= \phi_2 \rho \tau \psi \psi_M. && (79) \end{aligned}$$

where $\phi_2 \equiv (1/4)\,[\phi^3 + 2(1-\phi)^3 + 3(1-\phi)\phi^2]$. The structure of ϕ_2 can be derived by noting the covariance of the difference of averaged points of a Brownian motion with the difference of an averaged point and a particular point of the same Brownian motion. The result is left for the reader to verify.

Putting everything together, we have,

$$\begin{aligned} V_{L,M} &= \mathrm{Var}[(\hat{m}_t - \bar{m}_t) + (\bar{m}_t - m_i)], && (80) \\ &= \mathrm{Var}[\hat{m}_t - \bar{m}_t] + \mathrm{Var}[\bar{m}_t - m_i] + 2\ \mathrm{Cov}[\hat{m}_t - \bar{m}_t, \bar{m}_t - m_i], && (81) \\ &= \mathbf{S}_{(1,1)} + \phi_1 \tau \psi^2 + 2(\mathbf{G}_{(1,1)} - 1)\phi_2 \tau \psi^2 + 2\mathbf{G}_{(1,2)} \phi_2 \rho \tau \psi \psi_M, && (82) \end{aligned}$$