Quasi-arbitrage and Price Manipulation*

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ABSTRACT

In an environment where trading volume affects security prices and where prices are uncertain when trades are submitted, quasi-arbitrage is the availability of a series of trades which generate infinite expected profits with an infinite Sharpe ratio. We show that when the price impact of trades is time stationary, only linear price-impact functions rule out quasi-arbitrage and thus support viable market prices. This holds whether a single asset or a portfolio of assets is traded. When the temporary and permanent effects of trades on prices are independent, only the permanent price impact must be linear while the temporary one can be of a more general form. We also extend the analysis to a nonstationary framework.

IN ANY MARKET, TRADES can affect prices. In financial markets, the same individual can buy and subsequently sell the same security. In principle, then, a trader in a financial market can manipulate prices by buying and then selling the same security, with the expectation of earning a positive profit from such a manipulation.

This paper takes the perspective of a market watcher who has no opinion on the direction of security price movements but is an excellent student of the relation between trades and price changes. In fact, he has estimated that relation with absolute precision, and is tempted to exploit this knowledge to his advantage. In a market in which prices are uncertain when orders are submitted, he attempts to implement a *quasi-arbitrage* which is a trading strategy that produces infinite expected profits with an infinite Sharpe ratio. What are the possible relations between price changes and trades that rule out quasi-arbitrage for this market watcher?

The absence of quasi-arbitrage is tantamount to market viability unless all agents are too risk averse. A market is *viable* if no agent with a mean/standard-deviation utility function wishes to trade an infinite amount.

The dependence of price on trade size has an immediate as well as a permanent component. The *price-impact function* is the immediate price reaction to traded volume, including both temporary and permanent effects. The *price-update function* is the permanent effect of trade size on future prices. This paper's main result is the characterization of the price-update function under stationarity. Specifically, price-update functions that admit no quasi-arbitrage possibilities and thus ensure viable markets are linear in trade size.

Recent empirical papers assume in addition to stationarity that the price-impact and price-update functions are the same and suggest nonlinear price-update functions. Examples include Hasbrouck (1991), Hausman et al. (1992), and Kempf and Korn (1999). Interpreted in light of our work, these empirical results imply the feasibility of profitable manipulation. Alternatively, our work calls into question either the stationarity underlying much of the empirical work or the identification of the price-impact with the price-update function. (Holthausen et al. (1987), Gemmill (1996), or Keim and Madhavan (1996) make exactly the distinction between temporary and permanent effects of trades on prices.)

Anticipating some of this paper's results, Black (1995) imagines equilibrium exchanges where only limit orders labeled by levels of urgency are traded, and informally argues that price moves at each urgency level should be roughly proportional to order sizes to avoid "arbitrage". Presumably, he had in mind a time-stationary framework where trades have a permanent price impact only. Our paper does not address limit orders, but provides a formal proof of Black's conjecture when only market orders are allowed.

The standard justification of a price-impact function in an environment with asymmetrically informed agents is that information is impounded into prices through trades. Kyle (1985) is the leading example. Such models assume linear price-impact functions for tractability. This paper argues that linearity is justified in an environment that rules out quasi-arbitrage. It thereby selects which price-update and price-impact functions qualify for equilibrium.

The framework of this paper can also be used to evaluate Black's (1995) conjecture that the Kyle (1985) model allows price manipulation. We demonstrate that this is wrong for the monopolistic version of Kyle, but true for the multiple-insider version if the number of insiders or the number of trades is sufficiently large. Furthermore, we prove that the equilibrium price-impact functions in Kyle have to be linear when time between trades is small and only smooth price-impact functions are considered.

The recent incomplete-markets, asset-pricing literature suggests to impose bounds on either the Sharpe ratio (see Cochrane and Saa-Requejo (2000)) or the gain-loss ratio (see Bernardo and Ledoit (2000)) to rule out "good deals". Note that a costless investment opportunity that produces a Sharpe ratio of δ is referred to as δ-arbitrage in the literature (see Ledoit (1995)). These bounds are then used to calculate price bounds for the assets in the economy. Fortunately, the framework of this paper can also embed such restrictions on prices. All results derived here for the absence of quasi-arbitrage

(NoQA) are qualitatively the same as those for imposing no δ-arbitrage *(NoδA)* or no high gain-loss ratio.

Also, a simple corollary of our analysis is that ruling out price manipulation in a pure risk-neutral *(NoM)* world has the same implications for the shape of the price-update and price-impact functions as the absence of quasi-arbitrage. A *price manipulation* embodies a trading strategy that generates positive expected profits. However, note that a risk-neutral market is viable if and only if unbounded price manipulation is impossible *(NoUM)*.

The rest of the paper is organized as follows. Section I previews our results. Section II defines quasi-arbitrage and relates it to various notions of arbitrage which are used in the literature. Section III introduces the model and describes the properties of prices. Section IV characterizes the absence of quasi-arbitrage when the price-update and price-impact functions are stationary. Section V investigates nonstationary price-update and price-impact functions and discusses the Kyle (1985) model. Section VI treats multi-asset price dynamics. Section VII studies the relationship between price impact and the gain-loss ratio, and Section VIII concludes. All proofs are relegated to the Appendix.

I A Preview of the Results

The main results of this paper are formulated in ten propositions which offer conditions on the shape of the price impact functions that are necessary, sufficient, or equivalent to the absence of price manipulation, quasi-arbitrage, and market viability. The first seven propositions deal with time-stationary price-update and price-impact functions for one asset. Propositions 8-10 relax the assumption of stationarity and allow liquidity to vary across time, still in the single-asset framework. Since all ten propositions are also true for multi-asset price-update and price-impact functions, the multi-asset case requires only little additional analysis.

It is Proposition 7 that motivates all results in this paper. According to it, the absence of quasi-arbitrage is equivalent to the viability of markets, if agents are not too risk averse. As Dybvig and Ross

(1987) point out, arbitrage in complete markets cannot be allowed as it induces agents who prefer more to less to take on infinite positions. They actually demonstrate the equivalence between the absence of arbitrage and the existence of a competitive equilibrium. However, if markets are incomplete, this equivalence breaks down. For example, Loewenstein and Willard (2000) show that in the presence of credit constraints, an agent who prefers more to less may still be able to compute his optimal portfolio even though arbitrage is possible. Similarly, Basak and Croitoru (2000), Liu and Longstaff (2000), and Shleifer and Vishny (1997) prove the existence of equilibrium for markets in which arbitrage occurs but agents face portfolio constraints of various kinds. In view of this, Proposition 7 implies that the market incompleteness studied in this paper still allows a certain equivalence between market viability and the postulate of no arbitrage.

Propositions 1 and 2 assert that the price-update function must be linear in trade size to rule out quasi-arbitrage, regardless of the shape of the temporary price impact. Since Proposition 1 makes no assumptions about the distributions of the random elements of the price, such as news arrival and noise trades, the linearity of the price-update function holds only in expected terms. However, in Proposition 2, where all relevant random variables are normal, the linearity derived is exact.

This linearity contradicts empirical findings of a nonlinear price update. We will discuss a couple of regression equations used in the literature in Section IV.A. One argument often advanced to justify nonlinear price-update functions is that transaction costs outweigh the gains derived from exploiting nonlinear price updating. For example, if the price-update function is concave for purchases and convex for sales, as found in Hasbrouck (1991) and Kempf and Korn (1999), it would induce a quasi-arbitrageur to bid up the price by buying many small lots of shares over time and then to unwind his position by selling all the shares at once. Without transaction costs, such a strategy is profitable on average if sufficiently many trades are done. However, even in the presence of transaction costs, quasi-arbitrage is possible. This is the main implication of Propositions 1 and 2 which permit fixed and per-share transactions costs.

By virtue of Proposition 7, Propositions 1 and 2 are useful for choosing equilibrium price-update and price-impact functions, because market viability is essential for any equilibrium. Compare this with Allen and Gale (1992) who employ a Glosten and Milgrom-type (1985) framework to construct equilibria in which uninformed agents profitably manipulate the security price. However, this price manipulation does not erode the equilibrium because traders are volume-constrained.

Propositions 3 and 4 impose conditions on the random elements of the price and on the price-impact function, respectively. Proposition 3 contends that the absence of quasi-arbitrage rules out trends: the conditional expectation of noise trades and news must be zero. Otherwise, quasi-arbitrage would be feasible. Proposition 4 states two conditions that the price-impact function must respect in order to exclude quasi-arbitrage. First, the difference between the price impact of buying q shares and that of selling q shares must be no smaller than the price update of buying q shares. Second, the price-impact function cannot be constant unless the price-update function is zero. Both facts will become clearer following the model's introduction in Section III.

If the price-impact function is a multiple of the price-update function, then Proposition 5 shows that the absence of quasi-arbitrage is characterized by the linearity of both functions involved. Hence, linearity is also sufficient for the absence of quasi-arbitrage. For arbitrary price-impact functions and linear price-update functions, Proposition 6 provides a condition on the shape of the price-impact function that guarantees the infeasibility of quasi-arbitrage. This condition says that quasi-arbitrage is ruled out whenever the price-impact function is large enough relative to the price-update function.

Note that Jarrow (1992) investigates whether a large trader, whose trades move the price in an otherwise complete market, can make profits from price manipulation. He gives several examples of pure arbitrages and states a sufficient condition that rules out arbitrage, but is unable to characterize it. A characterization is quite difficult in his framework since very general price processes are permitted. However, our Proposition 5 demonstrates that for the prices studied here there exists an equivalent condition for market viability.

Proposition 8 considers nonstationary price-update and price-impact functions when they are linear and their slopes vary stochastically over time. It implies that the absence of price manipulation requires expected market liquidity, measured by the slopes of both the price-update and price-impact functions, not to decrease too fast. If all functions are restricted to be deterministic, then both the absence of quasi-arbitrage and the absence of price manipulation can even be characterized by conditions defined on the functions' slopes, as Proposition 9 claims.

Finally, Proposition 10 proves that the equilibrium price-impact functions in the Kyle (1985) model have to linear if time between trades becomes small.

II Definition of Quasi-arbitrage

In complete markets a pure arbitrage is defined as a costless, self-financing investment opportunity which renders a nonnegative payoff with probability one and a positive payoff with positive probability at some point later . More formally, suppose K assets can be traded over N periods and that all possible K-dimensional price vectors are described by the set $\mathcal{P}$, where each element of $\mathcal{P}$ is defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A trading strategy can be described by the sequence $(p_n, \theta_n, \mu_n)_{n=1}^N$, where $p_n \in \mathcal{P}$, θ_n denotes the vector of portfolio weights for the K assets, and μ_n is the cash held at time n. A pure arbitrage is a trading strategy which satisfies $p_1^T\theta_1 + \mu_1 = 0$ (no costs of investing), $p_n^T\theta_{n-1} + \mu_{n-1} = p_n^T\theta_n + \mu_n$ for $1 \leq n \leq N$ (self-financing property; note that the risk-free interest rate is assumed to be zero here for convenience), and $\mathbb{P}[p_N^T\theta_N + \mu_N \geq 0] = 1$ and $\mathbb{P}[p_N^T\theta_N + \mu_N > 0] > 0$. If we introduce the sequence of vectors $\{q_n\}_{n=1}^N$, where each q_n represents the (signed) traded quantities of the K assets at time n (purchases have a positive and sales have a negative sign), a pure arbitrage can be characterized as follows: there exists a pure arbitrage opportunity if and only if there exists a sequence of trades $\{q_n\}_{n=1}^N$ such that $\sum_{n=1}^N q_n = 0$, $\mathbb{P}[\sum_{n=1}^N p_n^T q_n \leq 0] = 1$, and $\mathbb{P}[\sum_{n=1}^N p_n^T q_n < 0] > 0$. In words, whenever there is a pure arbitrage one can find a multiperiod "round-trip" trading strategy which produces nonpositive trading costs with probability one and negative trading costs with positive probability; on the other hand, a trading strategy of the latter type is always a pure arbitrage.

A crucial assumption underlying complete markets is that prices are known before trades are initiated, that is, at the beginning of each period n, every trader observes the price vector p_n for which he can transact. If this condition is violated, pure arbitrages are hard to implement, because prices may fall after a trader submitted a sell order or rise after a buy order was initiated. Such adverse price movements make it difficult to avoid states in which losses occur with positive probability. Pure arbitrage is de facto infeasible when there is a chance that prices become zero.

This paper considers prices which are uncertain before trades take place. To determine the set of prices which support a viable market, we therefore cannot rely on a no pure-arbitrage condition but need to find conditions that take into account the specific incompleteness of the markets discussed here.

The basic object of our arbitrage concept is the set

$$\Pi \triangleq \Bigg\{ \pi_N^0 \in \mathbf{R} \mid \pi_N^0 = -\sum_{n=1}^{N} p_n^T q_n - c(N), \tag{1}$$

$$\text{where} \sum_{n=1}^{N} q_n = 0,\ q_n \in \mathcal{D}_M,\ p_n \in \mathcal{P},\ \text{for } 1 \leq n \leq N,\ N \in \mathbf{N} \Bigg\}$$

of all possible trading revenues caused by multiperiod round-trip trades ($\mathcal{D}_M$ denotes the domain of the trade size). We will refer to any element of Π simply as round-trip trade. Note that every round-trip trade in Π is self-financing. The function $c(.)$ measures the fixed costs of N transactions. For instance, if a commission fee of c is levied for each transaction, $c(N)$ would be cN (one could also make $c(.)$ depend on the number of assets traded within one transaction; but since this generalization would not add anything to our analysis we omit it). Having Π we can now formalize the idea of a price manipulation.

A *(risk-neutral) price manipulation* is a round-trip trade $\pi_N^0 \in \Pi$ with $\mathbb{E}[\pi_N^0] > 0$. Allen and Gale (1992a and 1992b) examine this type of price manipulation in the Glosten and Milgrom (1985) framework. This paper is based on the following definitions.

Definition 1 *An unbounded (risk-neutral) price manipulation is a sequence* $\{\pi_{N_m}^0\}_{m=1}^{\infty}$ *of round-trip* *trades with* $lim_{m\to\infty}\mathbb{E}[\pi^0_{N_m}] = \infty$.

Definition 2 *A quasi-arbitrage is an unbounded price manipulation* $\{\pi^0_{N_m}\}_{m=1}^{\infty}$ *which satisfies* $\lim_{m\to\infty}\frac{\mathbb{E}[\pi^0_{N_m}]}{Std[\pi^0_{N_m}]} = \infty$.

A quasi-arbitrage is thus a sequence of round-trip trades that exhibit not only infinite expected profits but also infinite expected profits per unit of risk, since the ratio $\text{SR}(\pi^0_{N_m}) \triangleq \mathbb{E}[\pi^0_{N_m}]/Std[\pi^0_{N_m}]$ can be interpreted as the "Sharpe ratio" of the trading profits. The standard deviation is allowed to converge to infinity as long as the expected value grows at a faster rate. Observe that our Sharpe ratio is homogeneous of degree zero. Therefore, if a quasi-arbitrage is *divisible* (meaning that any fraction of the quasi-arbitrage can be held), the risk can be reduced without altering the Sharpe ratio. In fact, the risk of a quasi-arbitrage can be eliminated asymptotically, because one can always find a sequence of portfolio weights $\{\theta_n\}_{n=1}^{\infty}$ such that $\lim_{m\to\infty}\mathbb{E}[\theta_m\pi^0_{N_m}] = \infty$ and $\lim_{m\to\infty} Std[\theta_m\pi^0_{N_m}] = 0$. We will provide examples of such risk-eliminating investment strategies in Section IV.A.

As previously mentioned, we will also use here the notion of a δ*-arbitrage*. In our setup, a δ-arbitrage is an element $\pi^0_N \in \Pi$ for which $\text{SR}(\pi^0_N) > \delta$. Obviously, a quasi-arbitrage is a δ-arbitrage and consequently imposing the absence of δ-arbitrage limits the set of admissible prices considerably more than the absence of quasi-arbitrage does.

Note that in general one cannot determine whether the absence of quasi-arbitrage restricts prices more than the absence of pure arbitrage. It is interesting, however, that the absence of pure arbitrage is a stronger condition than the absence of quasi-arbitrage when $\mathbb{P}$ has a finite support. This can be seen as follows. By Chebyshev's inequality, any quasi-arbitrage $\{\pi^0_{N_m}\}_{m=1}^{\infty}$ converges in probability to infinity, and hence there exists a round-trip trade π^0_N such that $\mathbb{P}[\pi^0_N > 0] = 1$. Since π^0_N is a pure arbitrage, we conclude that prices that rule out pure arbitrage also forbid quasi-arbitrage. The reverse, however, fails to be true.

Quasi-arbitrage is closely related to asymptotic arbitrage as introduced in Huberman (1982) for the APT. An asymptotic arbitrage is a sequence of zero-cost investments that produces an infinite average return in the limit, while the variance of the returns falls to zero. Translated in our language, an *asymptotic arbitrage* is a sequence $\{\pi^0_{N_m}\}_{m=1}^{\infty}$ of round-trip trades such that $\lim_{m\to\infty}\mathbb{E}[\pi^0_{N_m}] = \infty$ and $\lim_{m\to\infty} Std[\pi^0_{N_m}] = 0$. So, in particular, it is also a quasi-arbitrage. From the aforementioned follows therefore that a quasi-arbitrage is the same as an asymptotic arbitrage if the former is divisible. But in case of indivisibility, a quasi-arbitrage is not necessarily an asymptotic arbitrage. There is one important difference between aymptotic arbitrage here and in Huberman (1982). While in Huberman asymptotic arbitrage occurs when the number of the return's (cross-sectional) factors becomes infinite, asymptotic arbitrage here occurs when the total trading volume goes to infinity.

III Price Model and Market Conditions

The first subsection describes the price process, while the second cites the main assumptions used for our market model.

A Price Dynamics

Consider a trader of K assets over N periods. Each asset can be bought or sold via market orders at any time. In each period n, the initial price of each asset is given by the last price update for each asset summarized by the vector $\tilde{p}_n$. In the absence of uncertainty a trader has to pay a total of $[\tilde{p}_n + P_n(q_n)]^T q_n$ if he trades the vector of quantities q_n, and the initial prices for the next period will be $\tilde{p}_{n+1} = \tilde{p}_n + U_n(q_n)$. The *price-impact function* $P_n : \mathcal{D} \to \mathbf{R}^K$, $\mathcal{D} \subseteq \mathbf{R}^K$, measures the immediate price reaction of each asset to the trade q_n, including both the permanent and the temporary price impact. The *price-update function* $U_n : \mathcal{D} \to \mathbf{R}^K$, on the other hand, describes the trade's permanent impact on future prices. Hence, the temporary price impact is the difference $P_n - U_n$. The domain $\mathcal{D}$ of the trade size will typically be $\mathbf{R}^K$ or $\mathbf{Z}^K$, but more general sets are allowed. The minimal assumptions that $\mathcal{D}$ has to respect is $\{0, \pm 1\} \subseteq \mathcal{D}$ and $d \in \mathcal{D}$ implies $-d \in \mathcal{D}$. If these conditions are met, we call $\mathcal{D}$ a *symmetric domain.*

From the trader's perspective, other orders are random. In each period, all orders are submitted simultaneously. In addition, news that reveals value-relevant information arrives randomly. To incorporate both types of uncertainty, the price process is augmented with stochastic terms as follows. After the most recent trade q_{n-1} at the end of period $n-1$, the public news $\varepsilon_n \in \mathcal{D}_\varepsilon \subseteq \mathbf{R}^K$ ($\mathcal{D}_\varepsilon$ is symmetric) is revealed at the beginning of period n and the price is updated to $\tilde{p}_n$, taking into account both the last trade and the latest news. Since trading takes place only at the end of the period, the trader knows $\tilde{p}_n$ and ε_n before his trade in period n, but not the net order size of the other market participants described by η_n, which is taken from the symmetric domain $\mathcal{D}_\eta$. Note that the trader's (symmetric) domain will be denoted by $\mathcal{D}_M \subseteq \mathbf{R}^K$, so that $\mathcal{D} = \mathcal{D}_M + \mathcal{D}_\eta$, $\mathcal{D}$ being the domain of the price-update and price-impact functions. This structure gives rise to the following price dynamics:

$$\tilde{p}_n = \tilde{p}_{n-1} + U_{n-1}(q_{n-1} + \eta_{n-1}) + \varepsilon_n \tag{2}$$

$$p_n = \tilde{p}_n + P_n(q_n + \eta_n),$$

where p_n denotes the transaction price. The η_n's represent the residual trades over time, i.e., all orders other than those of the trader; they are *iid* with zero expected value. The ε_n's describe the disclosure of news through time, and are also *iid* random variables with zero mean, independent of the η_n's. Both stochastic processes are defined on the same probability space, $(\Omega, \mathcal{F}, \mathbb{P})$. (If the range of the random variables covers $\mathbf{R}^K$, negative prices cannot be excluded.) The zero means of η_n and ε_n imply that the prices in (2) form a martingale if zero net total trading volume is expected.

In view of (2), buying q_n costs $p_n^T q_n$ and the initial prices for the subsequent period are given by $\tilde{p}_{n+1}$. Moreover, note that the initial quote $\tilde{p}_n$ is the origin of the price-impact function in period n.

Since the trader knows ε_n but not η_n before his trade at time n, uncertainty over the current price is thus captured only by η_n, while uncertainty over subsequent prices is determined by the randomness of $\{\varepsilon_j\}_{j=n+1}^N$ and $\{\eta_j\}_{j=n}^N$. After the trade has occurred, the trader can extract η_n directly from the price p_n only when P_n is strictly monotonic; otherwise he must get the information on η_n from the publicly available records of trades at the exchange house. This environment should best capture real trading activity: while it is unlikely that new information occurs at the moment of submitting a trade, other trades not known to a trader are likely to happen. Summarizing, public information at time n, which we denote by $\mathcal{F}_n$, includes the knowledge of $\{\tilde{p}_j\}_{j=1}^{n}$, $\{p_j\}_{j=1}^{n-1}$, $\{\eta_j\}_{j=1}^{n-1}$, and $\{\varepsilon_j\}_{j=1}^{n}$. The conditional expectation given $\mathcal{F}_n$ will be denoted by $\mathbb{E}_n$. The price-update and price-impact functions are deterministic and therefore known.

We assume that in every period competitive liquidity providers stand ready to fill all the orders with a total volume of $q_n + \eta_n$. The prices given by (2) are thus set by those liquidity providers, with the price-update and price-impact functions representing their price reaction to trade size. Such providers resemble the market makers in Kyle (1985). In addition, fixed transactions costs may be charged by intermediaries: $c(k)$ specifies the fixed costs of k trades.

All trading takes place within the time interval $[0, 1]$ which represents a short-term time horizon like one day or one week. Hence, the riskless rate can be set to zero. Any number of trades or equivalently any frequency of trading is permitted. At time zero, before trading starts, each trader indicates his preference for the maximal time that should be allowed between trades. Since there are only finitely many traders, the exchange can choose a trading frequency that accommodates all traders preferences. This frequency is made public at time zero.

A relatively tractable special case of (2) is

$$\tilde{p}_n = \alpha \tilde{p}_{n-1} + (I_K - \alpha) p_{n-1} + \varepsilon_n \tag{3}$$

$$p_n = \tilde{p}_n + P_n(q_n + \eta_n)$$

where I_K is the K-dimensional identity matrix and $\alpha = diag(\alpha_1, \alpha_2, \ldots, \alpha_K)$, all $\alpha_j \in [0, 1]$. The price dynamics (3) can be obtained by setting $U_n = (I_K - \alpha)P_n$. For each asset, the individual faces an initial price that is a convex combination of the previous initial price and the price of the last trade. In this case, temporary and permanent price changes are closely linked. This will allow the derivation of stronger conditions that are implied by the absence of quasi-arbitrage or the absence of price manipulation. When $\alpha = 0$, i.e., $U_n = P_n$, then (3) simplifies to

$$p_n = p_{n-1} + U_n(q_n + \eta_n) + \varepsilon_n, \tag{4}$$

implying that the price change is a function of the current trade and randomness only, i.e., it does not depend on history. The recursion in (4) asserts that the transaction price and the price update for each asset coincide and that each trade has only a permanent impact on the security prices.

Observe that the Kyle (1985) model can be retrieved from (4). Just set $K = 1$ and $\mathcal{D} = \mathcal{D}_M = \mathcal{D}_\eta = \mathcal{D}_\varepsilon = \mathbf{R}$. Consequently, the price model in (2) should be understood as a generalization of the Kyle model.

B Market Classification

The variance-covariance matrices $Var[\eta_N]$ and $Var[\varepsilon_N]$ of the residual trades and news (if they exist) may depend on the total number of trades, N. We write $Var[\eta_N] = O(f(N))$ to express the fact that each (i,j)-component of $Var[\eta_N]$, as a function of N, asymptotically evolves as $f_{ij}(N)$, i.e., $\lim_{N\to\infty} Var[\eta_N]_{ij}/f_{ij}(N)$ is a positive constant, and we write $Var[\eta_N] = o(f(N))$ if $\lim_{N\to\infty} Var[\eta_N]_{ij}/f_{ij}(N) = 0$. In complete markets, typically $Var[\eta_N] = O(\frac{1}{N}1_{K\times K})$ is postulated, where $1_{K\times K}$ is the K-dimensional square matrix with only ones. Basically, what this says is that the total variance and covariances of each asset during a fixed time horizon is evenly divided between the N per-period variances and covariances of that asset. In contrast, we will permit more general behavior of the variances, for instance, we will also study the case $Var[\eta_N] = O(1_{K\times K})$, where the total variances and covariances accrued during $[0,1]$ are increasing linearly in the trading frequency. Such an assumption is appropriate if market volatility rises due to a higher trading intensity. Note that, in a strict sense, all terms appearing in (2) actually depend on N. Nevertheless, our notation will suppress this dependence unless necessary for clarity.

Fixed costs are zero or always satisfy $c(N) = O(N^{\varkappa})$, $\varkappa < 2$. We distinguish here four markets depending on the properties of the price uncertainty and the price-update and price-impact functions. To this end, define $\widehat{P}_n : \mathcal{D}_M \rightarrow \mathbf{R}^K \quad q \longmapsto \mathbb{E}[P_n(q+\eta)]$ and $\widehat{U}_n : \mathcal{D}_M \rightarrow \mathbf{R}^K \quad q \longmapsto \mathbb{E}[U_n(q+\eta)]$, the *expected price-impact function* and the *expected price-update function*, respectively. If all $\widehat{P}_n$'s and $\widehat{U}_n$'s exist, in other words, if prices all have first moments, then we call the market described by (2) $\mathcal{M}_1(\mathcal{P}_1, \Pi_1)$, where $\mathcal{P}_1$ is the set of prices and Π_1 is the corresponding set of round-trip trades (see (1)).

In the second market not only the $\widehat{P}_n$'s and $\widehat{U}_n$'s exist, but also $V(q; P_n, N) \triangleq Var[P_n(q + \eta_N)] > 0$ and $V(q; U_n, N) \triangleq Var[U_n(q + \eta_N)] > 0$ are well defined for all $q \in \mathcal{D}_M$, where η_N assumes the distribution of the residual trades if N is the maximal number of transactions. Furthermore, we require $V(q; P_n, N) = O(N^{\gamma} 1_{K \times K})$, $V(q; U_n, N) = O(N^{\zeta} 1_{K \times K})$, and $Var[\varepsilon_N] = O(N^{\vartheta} 1_{K \times K})$, where $\gamma < 1$, $\zeta < 1$, and $\vartheta < 1$. Hence, per-period variances and covariances of the price updates and price impacts can go up no more than linearly in N. Put differently, the total variances and covariances during the time interval $[0, 1]$ can grow no more than quadratically in N. The market that meets all conditions stated in this paragraph shall be labelled $\mathcal{M}_2(\mathcal{P}_2, \Pi_2)$.

$\mathcal{M}_1(\mathcal{P}_1, \Pi_1)$ and $\mathcal{M}_2(\mathcal{P}_2, \Pi_2)$ become the markets $\mathcal{M}'_1(\mathcal{P}'_1, \Pi'_1)$ and $\mathcal{M}'_2(\mathcal{P}'_2, \Pi'_2)$, respectively, if each satisfies in addition $\widehat{U}_n(q) \geq -\widehat{U}_n(-q)$ for all $q \in \mathcal{D}_M$, $q \geq 0$. This inequality says that purchases have an expected price update no smaller than sales. It can be interpreted in various ways. One argument is that sales often occur because of liquidity shocks and hence have less informational content. Or, the exchange may want to adopt such a rule to make price upwards movement more likely than downwards movement (there is a lower bound on prices, namely zero, but not an upper bound; moreover, if one embeds our framework into a long-run model, prices need an upwards trend to be attractive enough). Obviously, $\mathcal{M}'_2(\mathcal{P}'_2, \Pi'_2) \subseteq \mathcal{M}'_1(\mathcal{P}'_1, \Pi'_1) \subseteq \mathcal{M}_1(\mathcal{P}_1, \Pi_1)$ and $\mathcal{M}'_2(\mathcal{P}'_2, \Pi'_2) \subseteq \mathcal{M}_2(\mathcal{P}_2, \Pi_2) \subseteq \mathcal{M}_1(\mathcal{P}_1, \Pi_1)$.

The conditions introduced above are quite general. Here are some candidate price-update and price-impact functions and candidate distributions of residual trades and news that meet the requirements of our markets. For simplicity assume $K = 1$ and $\mathcal{D} = \mathcal{D}_M = \mathcal{D}_\eta = \mathcal{D}_\varepsilon = \mathbf{R}$. If the residual trades and news are normally distributed and (real) mappings like $q \longmapsto |q|^\beta \, sign(q)$, $0 < \beta < 1$, or $q \longmapsto q^{2j-1}$, $j \in \mathbf{N}$, are chosen for the price-update and price-impact function, then all three markets can be created by selecting appropriate variances as a function of N. In fact, any function dominated (in absolute value) by either of the two classes of mappings is a candidate when it exhibits moderate variation. Hence, candidate functions need not be smooth. Note, however, that candidate functions must be continuous, $L(\mathbf{R})$-*a.e.* ($L(A)$, $A \subseteq \mathbf{R}^K$, is the Lebesque measure on A), and bounded on each compact interval, if the residual trades have a continuous distribution. If it is discrete, not even continuity is required, though markets $\mathcal{M}_2(\mathcal{P}_2, \Pi_2)$ and $\mathcal{M}_2'(\mathcal{P}_2', \Pi_2')$ require the variation of the price-update and price-impact functions not to be too large. Whether the support of the residual trades' distribution (continuous or discrete) is bounded or unbounded does not affect the results derived below.

Naturally, we say that a market $\mathcal{M}(\mathcal{P}, \Pi)$ is free of quasi-arbitrage or price manipulation if there exist no quasi-arbitrages or price manipulations in the market, respectively, independent of the initial price.

A property of a function which is closely related to the absence of quasi-arbitrage and the absence of price manipulation (as we shall prove below) is *quasi-linearity.*

Definition 3 *A function $f : \mathcal{D} \to \mathbf{R}^K$ is quasi-linear if it has the representation*

$$f(y) = \lambda y + S_f(y) \tag{5}$$

on $\mathcal{D}$, $L(\mathcal{D}) - a.e.$, $\lambda \in \mathbf{R}^{K \times K}$ being positive semidefinite, where the $\mathcal{D}$-Borel-measurable function $S_f : \mathcal{D} \to \mathbf{R}^K$ satisfies

$$\mathbb{E}_n[S_f(\tilde{q}_n + \eta_n)] = 0 \tag{6}$$

for all $\mathcal{F}_n$-measurable random variables $\tilde{q}_n : \Omega \to \mathcal{D}$, $1 \leq n \leq N$. We call S_f the supplementary function of f.

We will interpret this definition after Proposition 1. Only note that equation (6) does not imply $S_f = 0$ in general (see Appendix B), and that in the case $K = 1$, positive semidefiniteness of λ boils down to λ being a nonnegative real number.

IV Single-asset Time-stationary Price Impact

The two examples below illustrate that price-update functions can cause quasi-arbitrage or bounded price manipulation. Both examples assume $U = P$, $\mathcal{D}_M = \{0, \pm 1, \pm 2, \pm 3\}$, $\mathcal{D}_\eta = \mathcal{D}_\varepsilon = \{0, \pm 1, \pm 2\}$, $\mathbb{P}[\eta_n = 0] = \frac{2}{5}$, $\mathbb{P}[\eta_n = 1] = \mathbb{P}[\eta_n = -1] = \frac{1}{5}$, $\mathbb{P}[\eta_n = 2] = \mathbb{P}[\eta_n = -2] = \frac{1}{10}$, and $p_0 = 41\lambda$, $\lambda > 0$.

Example 1. Consider the price-update function

$$U(x) = \begin{cases} 0 & \text{if } x = 0 \\ \pm 2\lambda & \text{if } x = \pm 1 \\ \pm 4\lambda & \text{if } x = \pm 2 \\ 10\lambda & \text{if } 3 < x < 5 \\ 5\lambda & \text{if } -5 < x < -3 \end{cases},$$

which satisfies $\mathbb{E}[U(\eta_n)] = 0$ (the trader does not anticipate a price change unless he trades). In this case, purchases move the price more than sales. Certain empirical papers report such an asymmetric price impact in that block purchases have a larger price impact than block sales (see, e.g., Gemmill (1996) or Holthausen et al. (1987)). Chan and Lakonishok (1995) report the same for institutional trades. Such a price-update function allows quasi-arbitrage. The trading strategy of buying three units in each of the first m periods and then selling three units in each of the following m period yields expected profits of

$$\mathbb{E}[\pi^0_{2m}] = \mathbb{E}[-\sum_{n=2}^{2m} U(q_n + \eta_n) \sum_{j=n}^{2m} q_j - \sum_{n=2}^{2m} \varepsilon_n \sum_{j=n}^{2m} q_j] = \frac{\lambda}{4} m(21m - 75)$$

because $\mathbb{E}[U(3+\eta_n)] = 8\lambda$ and $\mathbb{E}[U(-3+\eta_n)] = -\frac{9}{2}\lambda$. Note that $\mathbb{E}[\pi^0_{2m}] > 0$ when $m \geq 4$. A similar computation reveals that $Var[\pi^0_{2m}] = o(m^{3/2})$. Hence, $\mathrm{SR}(\pi^0_{2m}) \to \infty$ as $m \to \infty$.

Example 2. Keim and Madhavan (1996) and Scholes (1972) provide evidence that there are also markets with a stronger permanent price impact of sales. Such a situation arises if

$$U(x) = \begin{cases} 0 & \text{if } x = 0 \\ \pm 2\lambda & \text{if } x = \pm 1 \\ \pm 4\lambda & \text{if } x = \pm 2 \\ 5\lambda & \text{if } 3 < x < 5 \\ -10\lambda & \text{if } -5 < x < -3 \end{cases} .$$

(again $\mathbb{E}[U(\eta_n)] = 0$). Now, consider the strategy of selling two units in each of the first six periods, and then buying three units in each of the following four periods. Since $\mathbb{E}[U(-2+\eta_n)] = -5\lambda$ and $\mathbb{E}[U(3+\eta_n)] = \frac{9}{2}\lambda$, this strategy renders an expected profit of 15λ. Note that the number of trades, N, is uniformly bounded for all $\pi^0_N \in \Pi_1$, i.e., $\sup\{N \in \mathbf{N} \mid \exists N \text{ such that } \pi^0_N \in \Pi_1\} < \infty$, due to the nonnegativity of prices and the stronger impact of sales. Therefore, unbounded price manipulation and, in particular, quasi-arbitrage does not exist. A δ-arbitrage may exist depending on the level of δ.

A Necessary Conditions for the Absence of Quasi-arbitrage

We start with the main result of this subsection.

Proposition 1 *Each of the following conditions,*

(NoQA) the absence of quasi-arbitrage in $\mathcal{M}'_2(\mathcal{P}'_2, \Pi'_2)$*;*

(NoUM) the absence of unbounded price manipulation in $\mathcal{M}'_1(\mathcal{P}'_1, \Pi'_1)$*;*

(NoδA) the absence of δ-arbitrage in $\mathcal{M}_2(\mathcal{P}_2, \Pi_2)$*;*

(NoM) the absence of price manipulation in $\mathcal{M}_1(\mathcal{P}_1, \Pi_1)$*;*

requires U to be quasi-linear.

Proposition 1 says that each of (NoQA)-(NoM) implies a price-update function that can be written as the sum of a linear function (with nonnegative slope) and its supplementary function S_U of U, which in conditional expected terms drops out. The latter holds regardless of what order the trader submits, because the trader's strategy set is identical to the set consisting of all $\mathcal{F}_n$-measurable random variables. Hence, traders always expect linear price updating.

The intuition underlying Proposition 1 transpires from the outline of the proof offered below. For convenience, we divide the outline of the proof into four steps and assume $\mathcal{D} = \mathcal{D}_M = \mathcal{D}_\eta = \mathcal{D}_\varepsilon = \mathbf{R}$. The formal proof is in Appendix A. We only discuss (NoQA) and (NoM), because the results for the other two are consequences of the former. Also note that whenever we compute second moments or Sharpe ratios we implicitly assume that the conditions in $\mathcal{M}_2'(\mathcal{P}_2', \Pi_2')$ are met.

Step 1: The expected price-update function must be symmetric, i.e., $\hat{U}(q) = -\hat{U}(-q)$. To show this, note that either $\hat{U}(q) > -\hat{U}(-q)$ or $\hat{U}(q) < -\hat{U}(-q)$ for a $q > 0$ would invite price manipulation. In the former case (where purchasing q units has a stronger impact on the price update than selling q units), a trader could buy q shares in each of the first m periods and then sell q shares in each of the subsequent m periods. The mean and the Sharpe ratio of this round-trip strategy satisfy $\mathbb{E}[\pi_{2m}^0] = O(m^2)$ and $\lim_{m\to\infty}\mathrm{SR}(\pi_{2m}^0) = \infty$. In the second case, the reverse strategy (first selling q shares in each of the first m periods and then buying back q shares in each of periods $m+1$ to $2m$) would yield $\mathbb{E}[\pi_{2m}^0] > 0$ (note that $\hat{U}(q) < -\hat{U}(-q)$ need not be treated for (NoQA)). Hence, the symmetry of $\hat{U}$. It is straightforward to check that also $\hat{U}(q) \geq 0$ for $q > 0$ must hold.

Step 2: $\hat{U}$ is continuous, i.e., $\lim_{n\to\infty}\hat{U}(q_n^{(q)}) = \hat{U}(q)$, $q \in \mathcal{D}_M\backslash\{0\}$, when $\lim_{n\to\infty} q_n^{(q)} = q$. To sketch the idea of this part of the proof, consider the following example. Suppose that the price-update function has an upward jump at $q > 0$, that is, $\lim_{q_n^{(q)}\to q+}\hat{U}(q_n^{(q)}) > \hat{U}(q)$. The strategy of buying $q_n^{(q)} > q$ shares in each of the first m periods and selling q shares in each of the following m periods, where $q_n^{(q)}$ is chosen arbitrarily close to q, yields $\mathbb{E}[\pi_{2m}^0] = O(m^2)$ and $\lim_{m\to\infty}\mathrm{SR}(\pi_{2m}^0) = \infty$. Due to the jump, the updating reacts less to sales than to buys, causing the average selling price to exceed the average purchasing price. Appendix A demonstrates that for any possible type of jump there exists a quasi-arbitrage.

Step 3: $\hat{U}$ is linear, since each of (NoM) and (NoQA) is incompatible with $\hat{U}(q) > \hat{U}(1)q$ or $\hat{U}(q) < \hat{U}(1)q$, for an arbitrary q. To see this, consider the first case and note that $q > 0$ can be assumed to be a rational number. Now, buying q shares in each of the first m periods and then selling one share in each of the following mq periods (mq can be chosen to be an integer) yields $\mathbb{E}[\pi^0_{2m}] = O(m^2)$ and $\lim_{m\to\infty}\mathrm{SR}(\pi^0_{2m}) = \infty$, since the selling moves the price down by less than the degree to which the buying shifts the price upwards. The second inequality can be rejected analogously.

Step 4: Proving that U is quasi-linear. For this purpose, define $S_U(q) \triangleq U(q) - \hat{U}(q)$. Then, Step 3 implies $\mathbb{E}[S_U(q + \eta_n)] = 0$ for all q, which in turn has (5) and (6) for U as a consequence.

The quasi-arbitrages above always exhibit $Std[\pi^0_{2m}] = o(m^\theta)$, $\theta < 2$. As a consequence, the sequence of investments $\{m^{-\phi}\pi^0_{2m}\}^\infty_{m=1}$, $\theta < \phi < 2$, would constitute an asymptotic arbitrage, if the quasi-arbitrages were divisible.

Next, we propose two distributions of the residual trades, each of which causes the supplementary function of U to be zero. One possibility is that the residual trades are zero, and the second is that they are normally distributed.

Proposition 2 *Suppose* $\mathcal{D} = \mathcal{D}_M = \mathcal{D}_\eta = \mathcal{D}_\varepsilon = \mathbf{R}$ *and that either*

i. $\mathbb{P}[\eta_n = 0] = 1$, *for* $1 \le n \le N$ *(zero residual trades) or*

ii. the residual trades are normally distributed.

Then, the price-update function in Proposition 1 is linear, $L(\mathbf{R})-$*a.e., and not only quasi-linear.*

Notice that case *i* describes the situation where only one trader affects the price in each period ($\eta_n = 0$ means that the total net trading volume of all the other traders is zero). Contrary to Proposition 1, the absence of either price manipulation or quasi-arbitrage now requires the price-update function to be exactly linear and not only linear in expected terms. It can also be shown that Proposition 2 is true when the residual trades are a certain transform of a zero-mean normal random variable (for details see Remark 1 in Appendix A).

One important formal feature of the price process (2) is that the price-impact function P can be chosen to include fixed per-share transaction costs. Hence, Propositions 1 and 2 are also valid when commissions have to be paid per share.

Note that the supplementary function of U need not be zero for other distributions, as three examples in Appendix B demonstrate. For empirical studies, then, nonlinear price-update functions can be used, but their conditional expectation must be linear in trade size.

Proposition 2 provides a theoretical justification for looking at linear additive price processes of the type $p_n = p_{n-1} + \lambda q_n + \varepsilon_n$ (i.e., setting $U = P$), which has been popular in the literature, with tractability being the main motivation (see Dutta and Madhavan (1995), Hausman et al. (1992), or Bertsimas and Lo (1998)). Note that this specification is also sufficient to rule out price manipulation and quasi-arbitrage. This is one of the main results in the next section.

To assume $\widehat{U}(0) \neq 0$ or $\mathbb{E}[\varepsilon_n] \neq 0$ would be harmful in this context. For example, if $\mathbb{E}[\varepsilon_n] > 0$, then buying one share in the first period and selling this share some periods later would be profitable, because the price moves up between the purchase and the sale due to $\mathbb{E}[\varepsilon_n] > 0$. To exclude this kind of manipulation, the price process (2) must not include trend components in the short run. This justifies our zero-mean assumptions, and is stated as the next proposition.

Proposition 3 *If either $\mathbb{E}[U(0)] \neq 0$ or $\mathbb{E}[\varepsilon_n] \neq 0$, then the price process (2) will violate (NoM) and (NoδA). In particular, if $\mathbb{E}[U(0)] > 0$ or $\mathbb{E}[\varepsilon_n] > 0$, then each of (NoQA)-(NoM) does not hold.*

We can also derive necessary conditions for the price-impact function, although they have a less compact form than the conditions in Propositions 1 and 2. Thus, we are content here with giving only two of them.

Proposition 4 *If either (NoQA), (NoUM), (NoδA), or (NoM) is true, then the following two conditions must hold:*

i. $\hat{P}(q) - \hat{P}(-q) \gtreqless \hat{U}(q)$ *for* $q \gtreqless 0$, $q \in \mathcal{D}_M$, *and*

ii. P *cannot be constant when* $U \neq 0$.

If we interpret the left-hand side of condition i in Proposition 4 as the spread of the price-impact function, then condition i says that the spread *at any* trade size has to exceed the price update resulting from that trade. Were this not true, the trading strategy cited in Step 1 of the proof following Proposition 1 (buying q shares in each of the first m periods and then selling q shares in each of the next m periods) would allow price manipulation and quasi-arbitrage. The same trading strategy also implies the second condition in Proposition 4. P always has to be a function of the trade size, unless $U = 0$. Stated differently, price update and price impact can never offset each other perfectly, unless $U = P = 0$.

Proposition 2 rules out various tempting functional forms for the price-impact function. For instance, Breen et al. (2000) estimate a price-impact function where the inter-transaction return is linear in the traded quantity, i.e., $(p_n - p_{n-1})/p_{n-1} = \alpha + \lambda q_n + \varepsilon_n$. This regression equation implies $p_n = (1 + \alpha + \lambda q_n + \varepsilon_n)p_{n-1}$, giving rise to quasi-arbitrage. To see this, take $N = 2$ and $q_2 = -q_1$, and compute $\mathbb{E}[\pi_2^0] = p_0 q_1(-\alpha + \lambda q_1)(1 + \alpha + \lambda q_1) = O(q_1^3)$ and $\lim_{q_1 \to \infty} \mathrm{SR}(\pi_2^0) = \infty$. A second price process is $p_n = p_{n-1} + \lambda \frac{1}{n} \sum_{i=0}^{n-1} q_{n-i} + \varepsilon_n$ (Dutta and Madhavan (1995) employ a variant of this price process). It is easy to find a quasi-arbitrage for these prices. Finally, for price processes where the average is taken over prices rather than over the trading quantities, like $p_n = \sum_{i=1}^{m} \alpha_i p_{n-i} + \lambda q_n + \varepsilon_n$, $m > 0$ (see Hasbrouck (1991)), quasi-arbitrage or mere bounded price manipulation may be feasible, too, depending on the value of the α_i's.

B A Sufficient Condition for the Absence of Quasi-arbitrage

This subsection derives a sufficient condition for (NoQA)-(NoM). With the aid of this condition we are able to establish a characterization of the absence of price manipulation and quasi-arbitrage for the case $U = (1-\alpha)P$, where the price-update and price-impact functions are multiples of each other. The absence of quasi-arbitrage is equivalent to the linearity of both the price-update and price-impact functions. If U and P are independent, our sufficient condition will serve us to find some interesting examples of price-impact functions that give rise to market prices which do not allow quasi-arbitrage.

The main observation leading to this sufficient condition is the fact that $\sup\{\mathbb{E}[\pi_N^0] \mid \pi_N^0 \in \Pi\} = 0$ if U and P are both are quasi-linear and $P > \frac{1}{2}U$ for $q > 0$, as is shown in Appendix A. Hence, we can state the following auxiliary result.

Lemma 1 *Each of (NoQA)-(NoM) holds, whenever U and P are both quasi-linear and $P(x) \geq \frac{1}{2}U(x)$ for $x \geq 0$, $x \in \mathcal{D}$.*

Then, Lemma 1 and Propositions 1 and 2 imply the following.

Proposition 5 *Suppose that $U = (1-\alpha)P$. Then, quasi-linearity of U and P is equivalent to each*

(NoQA) the absence of quasi-arbitrage in $\mathcal{M}_2'(\mathcal{P}_2', \Pi_2')$;

(NoUM) the absence of unbounded price manipulation in $\mathcal{M}_1'(\mathcal{P}_1', \Pi_1')$;

(NoδA) the absence of δ-arbitrage in $\mathcal{M}_2(\mathcal{P}_2, \Pi_2)$;

(NoM) the absence of price manipulation in $\mathcal{M}_1(\mathcal{P}_1, \Pi_1)$.

If the residual trades assume one of the distributions stated in Proposition 2, then we obtain the stronger result that each of (NoQA)-(NoM) is characterized by the linearity of U and P.

Proposition 5 connects (NoQA)-(NoM) through one common characterizing property, namely, the quasi-linearity of the price-update and price-impact functions, provided the price evolves according to (3) or (4).

If P is not a multiple of U, then nonlinear price-impact functions can also lead to prices without quasi-arbitrage opportunities. With the help of the proposition below, which follows directly from Lemma 1, we can construct examples illustrating this point.

Proposition 6 *Let U be quasi-linear. If $P(x) \geq \frac{1}{2}U(x)$ for $x \geq 0$, $x \in \mathcal{D}$, and $P(x) \leq \frac{1}{2}U(x)$ for $x < 0$, $x \in \mathcal{D}$, then (NoQA)-(NoM) are all satisfied.*

Consider the price-impact function $P(x) = \frac{1}{2}[A\ sign(x) + U(x)]$, where $A > 0$ is a constant. This function exhibits a discontinuity at zero (with jump size A) and intersects the price-update function twice. From Proposition 6, both price manipulation and quasi-arbitrage are infeasible if U is quasi-linear. Glosten (1994) constructs an equilibrium with an open limit order book and describes situations where price revision intersects the actual price schedule.

Hasbrouck (1991) provides empirical evidence that security prices are concave for purchases and convex for sales. This relation can be modeled here by taking a symmetric price-impact function that is concave in some positive range without violating the conditions stated in Proposition 6, which imply that P has to grow (decline) at least linearly eventually.

Evidently, many more admissible U and P with very complicated price-impact functions can be found here. This suggests that in the case $U \neq (1-\alpha)P$, sufficient conditions for (NoQA)-(NoM) that are also necessary may be very hard to derive. We refrain from further examination.

C Market Viability

As defined in the introduction, a market is viable if and only if no agent wishes to trade an infinite amount of shares over a finite time horizon. If the utility is derived from the mean and standard deviation of an investment, then considering the agent with the smallest level of risk aversion is sufficient to verify a market's viability. The indifference curve $\mathcal{I}_{\overline{u}} : \mathbf{R}_+ \to \mathbf{R}$ of this trader is implicitly defined by $u(\mathcal{I}_{\overline{u}}(Std), Std) = \overline{u}$, $\overline{u} \in \mathbf{R}$. He is said to have *asymptotically moderate risk aversion* if $\mathcal{I}_{\overline{u}}(x) = O(x^{\theta})$ for $\theta < \frac{1}{2}(3-\psi)$, where $\psi = \max(\gamma, \zeta, \vartheta)$, $V(q; P_n, N) = O(N^{\gamma})$, $V(q; U_n, N) = O(N^{\zeta})$, and $Var[\varepsilon_N] = O(N^{\vartheta})$, $\gamma < 1$, $\zeta < 1$, and $\vartheta < 1$. Examples of such preferences are drawn in Figure 1. If $\psi = -1$ (classical case), the steepest permissible indifference curve is quadratic, like $\mathcal{I}_1$ in Figure 1. If $\psi = 0$ (constant per-period volatility of residual trades and news), the admissible indifference curves can grow as fast as $Std^{3/2}$. Finally, for large ψ close to one, they are asymptotically bounded below by linear functions. Hence, our assumptions always allow linear and decreasing indifference curves.

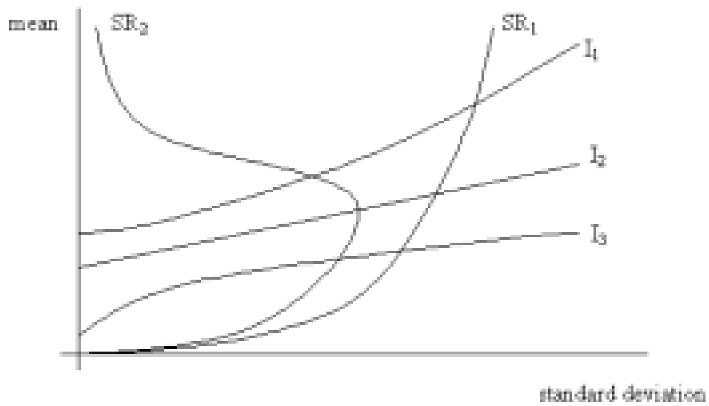

Figure 1: Sharpe-ratio frontiers and indifference curves

One may interpret strictly increasing and strictly convex indifference curves (like $\mathcal{I}_1$) as representing increasing risk aversion in the $(\mathbb{E}, Std)$ space. Similarly, strictly increasing indifference curves describe constant risk aversion when they are linear (see $\mathcal{I}_2$), and decreasing risk aversion, when they are concave (see $\mathcal{I}_3$). But note that $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$ would all exhibit nonincreasing absolute risk aversion (in the usual sense) if the round-trip trades π_N^0 were normally distributed and the utility were negative exponential. The indifference curve resulting from $\mathbb{E}[-e^{-\rho\pi_N^0}] = \overline{u}$, $\rho > 0$ being the risk aversion coefficient, would read $\mathcal{I}_{\overline{u}}(x) = \frac{\ln(-\overline{u})}{\rho} + \frac{\rho}{2}x^2$.

Figure 1 also depicts two examples of the Sharpe ratio frontier that can be generated by round-trip trades. SR_1 shows the case where quasi-arbitrage is indivisible, while SR_2 represents a divisible quasi-arbitrage. Both frontiers intersect the indifference curves shown for any level of $\overline{u}$. We therefore can conclude the main result on market viability.

Proposition 7 *If the agent with least risk aversion is risk neutral, then $\mathcal{M}_1'(\mathcal{P}_1', \Pi_1')$ is viable if and only if there exists no unbounded price manipulation. If the same agent exhibits asymptotically moderate risk aversion, then $\mathcal{M}_2'(\mathcal{P}_2', \Pi_2')$ is viable if and only if quasi-arbitrage is infeasible. The last equivalence* *holds for any level of risk aversion of the agent when every quasi-arbitrage is divisible.*

By virtue of Proposition 1, the viability of $\mathcal{M}_2'(\mathcal{P}_2', \Pi_2')$ has the quasi-linearity of the price-update function as a consequence. For the case $U = (1-\alpha)P$ we obtain the even stronger finding that $\mathcal{M}_2'(\mathcal{P}_2', \Pi_2')$'s viability is tantamount to the quasi-linearity of U and P. This is immediate from Proposition 5.

Observe that Proposition 7 does not include a characterization of $\mathcal{M}_1(\mathcal{P}_1, \Pi_1)$'s viability. Trivially, $\mathcal{M}_1(\mathcal{P}_1, \Pi_1)$ is viable if and only if $s \triangleq \sup\{\mathbb{E}[\pi_N^0] \mid \pi_N^0 \in \Pi_1\}$ is not attained by any round-trip trade $\pi_N^0 \in \Pi_1$ and there exists a sequence $\{\pi_{N_m}^0\}_{m=1}^{\infty}$ with $\mathbb{E}[\pi_{N_m}^0] \to s$ and $\max_{1\leq n\leq N} |q_n| \to \infty$, as $m \to \infty$. Since most models in the microstructure literature focus on symmetric price-update and price-impact functions, we will not investigate further the implications of viability in $\mathcal{M}_1(\mathcal{P}_1, \Pi_1)$.

Proposition 7 can be directly compared to the two pivotal optimization problems studied in microstructure. The first involves insider trading, where a monopolistic insider solves $\sup_{\{q_n\}_{n=1}^N} \mathbb{E}[\sum_{n=1}^N (v - p_n)q_n]$ for a given N, after having received the value of the asset, v, in period 0. A thorough examination of this problem for a simpler variant of the price process (4) can be found in Dutta and Madhavan (1995). It is straightforward to show that the insider problem in market $\mathcal{M}_2'(\mathcal{P}_2', \Pi_2')$ ($\mathcal{M}_1'(\mathcal{P}_1', \Pi_1')$) has a solution if and only if there is no quasi-arbitrage (no unbounded price manipulation), or equivalently, $\mathcal{M}_2'(\mathcal{P}_2', \Pi_2')$ ($\mathcal{M}_1'(\mathcal{P}_1', \Pi_1')$) is viable.

The second problem that has been of interest is discretionary liquidity trading, as studied in Bertsimas and Lo (1998) and Huberman and Stanzl (2001). There, an uninformed trader faces the problem $\inf_{\{q_n\}_{n=1}^N} \mathbb{E}[\sum_{n=1}^N p_n q_n]$ subject to $\sum_{n=1}^N q_n = \overline{q} \neq 0$, given the number of trades, N. In other words, this trader wants to minimize the expected costs of trading a certain amount of shares, $\overline{q}$, over a certain discrete time horizon. Obviously, in market $\mathcal{M}_2'(\mathcal{P}_2', \Pi_2')$ ($\mathcal{M}_1'(\mathcal{P}_1', \Pi_1')$), there exists an optimal trading strategy for the liquidity trader if and only if $\mathcal{M}_2'(\mathcal{P}_2', \Pi_2')$ ($\mathcal{M}_1'(\mathcal{P}_1', \Pi_1')$) is viable.

V Nonstationary Price Impact

Until now, the price-update and price-impact functions have been time-stationary, i.e., price reacts to traded quantity in the same manner in each period. *Liquidity*, which is represented by the first derivative of the price-update and price-impact functions (when they exist), is therefore constant through time. In what follows we relax this assumption and allow liquidity to vary across time.

A Absence of Quasi-arbitrage

One way to examine nonstationary liquidity is to consider linear price-update and price-impact functions that change over time. More specifically,

$$\tilde{p}_n = \tilde{p}_{n-1} + \lambda_{n-1}(q_{n-1} + \eta_{n-1}) + \varepsilon_n \tag{7}$$

$$p_n = \tilde{p}_n + \mu_n(q_n + \eta_n),$$

where the sequences of random variables $\{\lambda_n : \Omega \to \mathbf{R}\}_{n=1}^N$ and $\{\mu_n : \Omega \to \mathbf{R}\}_{n=1}^N$ are assumed to be independent across time as well as of each other and all other uncertainty in this model; in addition, each λ_n and μ_n is $\mathcal{F}_n$-measurable, $\mu_1 \geq 0$ $\mathbb{P} - a.e.$, and for convenience we set $\hat{\lambda}_1 = \mathbb{E}[\lambda_1] = \mathbb{E}[\mu_1] = \hat{\mu}_1 \geq 0$, and the fixed-transaction cost function is $c(.) = 0$.

For the prices in (7) we will use the same market classification as introduced above. Only note that in contrast to the previous section we need to consider here only markets $\mathcal{M}_1'(\mathcal{P}_1', \Pi_1')$ and $\mathcal{M}_2'(\mathcal{P}_2', \Pi_2')$, since price-update and price-impact functions are symmetric by assumption. We will only discuss here the consequences of the absence of quasi-arbitrage and price manipulation.

We proceed by first establishing a necessary condition for the absence of price manipulation, and then we characterize the absence of quasi-arbitrage and price manipulation for the case of deterministic price-update and price-impact functions. In the next subsection we relate our results to the extant literature.

The main difference with Section IV is that both price manipulation and quasi-arbitrage can usually be implemented with finitely many trades. Unlike in the last section, we can employ the global shape of the price-update and price-impact functions to seek for quasi-arbitrage or price manipulation.

Let us first provide an example of price-update functions that permit quasi-arbitrage. Consider $p_0 = 10$, $\{\lambda_n\}_{n=1}^3 = \{\mu_n\}_{n=1}^3$ with $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 0$, $\mathbb{P} - a.e.$ Buying q units of the asset in each of periods 1 and 2, and then selling the holdings in the third period results in expected profits of $\mathbb{E}[\pi_3^0] = -(10+q)q - (10+2q)q + 2q(10+2q) = q^2$, while $Std[\pi_3^0] = \sqrt{Var[\eta] + 5Var[\varepsilon]}q$. Hence, $\mathbb{E}[\pi_3^0] \to \infty$ and $\mathrm{SR}[\pi_3^0] \to \infty$ as $q \to \infty$. This quasi-arbitrage requires only three trades but ever increasing amounts of shares.

To get an idea what kind of restrictions the absence of price manipulation imposes on the pair $(\{\lambda_n\}_{n=1}^N, \{\mu_n\}_{n=1}^N)$, let us consider the simple case $N = 3$ where only three trades are feasible.

Computing $\mathbb{E}[\pi_3^0]$ for deterministic trades leads to the quadratic form $-[\hat{\mu}_2 q_2^2 + \hat{\lambda}_2 q_2 q_3 + \hat{\mu}_3 q_3^2]$, or in matrix notation, $\mathbb{E}[\pi_3^0] = -\frac{1}{2}[q_2\ \ q_3]\Lambda_{3,-1}[q_2\ \ q_3]^T$, where

$$\Lambda_{3,-1} \triangleq \begin{bmatrix} 2\hat{\mu}_2 & \hat{\lambda}_2 \\ \hat{\lambda}_2 & 2\hat{\mu}_3 \end{bmatrix}.$$

(The term $p_0 + \hat{\lambda}_1 q_1$ drops out because it is a price component in each period and is thus canceled by $\sum_{n=1}^3 q_n = 0$.) The trades q_2 and q_3 in the above expression can take any value as long as the trades q_1, q_2, and q_3 do not cause negative expected prices. More formally, $\overrightarrow{q}_3 \in \mathfrak{P}_3$, where $\overrightarrow{q}_N \triangleq (q_1,\ q_2, \ldots, q_N)$ (so $\overrightarrow{q}_N^n = q_n$) and

$$\mathfrak{P}_N \triangleq \{\overrightarrow{q}_N \in \mathcal{D}_M^N \mid \mathbb{E}[p_n] \geq 0,\ \overrightarrow{q}_N^n \text{ is } \mathcal{F}_n \text{ measurable}, 1 \leq n \leq N,\ \sum_{n=1}^N \overrightarrow{q}_N^n = 0,\ N \in \mathbf{N}\}.$$

Observe that $\mathfrak{P}_N$ is a system of linear inequalities and hence polyhedral convex.

Evidently, whenever there exists a vector $\overrightarrow{q}_{3,-1}$, $\overrightarrow{q}_{N,-1} \triangleq (q_2, q_3, \ldots, q_N)$, such that $\overrightarrow{q}_{3,-1}^T \Lambda_{3,-1} \overrightarrow{q}_{3,-1} < 0$, then there exists a sequence of points $\theta_h(-q_2 - q_3, q_2, q_3)$, $\theta_h \to 0$, having the same property. Since

0 is an interior point of $\mathfrak{P}_3$, we therefore conclude that $\Lambda_{3,-1}$ has to be positive semidefinite to rule out price manipulation.

The removal of q_1 is arbitrary. If we remove q_2 or q_3, we would find that $\Lambda_{3,-2} \triangleq \begin{bmatrix} 2\hat{\mu}_2 & \hat{\lambda}_2 \\ \hat{\lambda}_2 & 2\hat{\mu}_3 \end{bmatrix}$ and $\Lambda_{3,-3} \triangleq \begin{bmatrix} 2\hat{\mu}_3 & 2\hat{\mu}_3 - \hat{\lambda}_2 \\ 2\hat{\mu}_3 - \hat{\lambda}_2 & 2\hat{\mu}_3 \end{bmatrix}$, respectively, have to be positive semidefinite to exclude price manipulation. Since all matrix representations employ the same parameters and each matrix is positive semidefinite if and only if the others are, either matrix can be used for the analysis. We work here only with $\Lambda_{3,-1}$ henceforth.

For $\Lambda_{3,-1}$ to be positive semidefinite, $\hat{\mu}_2$ and $\hat{\mu}_3$ must be nonnegative and $\hat{\mu}_2\hat{\mu}_3 \geq \hat{\lambda}_2^2/4$. These conditions, together with $\hat{\mu}_1 \geq 0$, say that the absence of price manipulation rules out negative (expected) price-impact sequences in all periods and that $\hat{\mu}_2$ and $\hat{\mu}_3$ have to be sufficiently large relative to $\hat{\lambda}_2^2$. Notice that $\hat{\lambda}_2$ can be negative, conflicting with the interpretation that purchases convey positive news about the asset's value and push the price up. We will discuss this issue below when we have at our disposal a sufficient condition for the absence of price manipulation.

The same method as above applied to the general case gives the following.

Proposition 8 *If the price-update and price-impact slopes are random, then the absence of price manipulation in $\mathcal{M}_1'(\mathcal{P}_1', \Pi_1')$ implies that the expected value $\hat{\Lambda}_N$ of the matrix Λ_N defined by*

$$\Lambda_N \triangleq \begin{bmatrix} 2\mu_2 & \lambda_2 & \lambda_2 & \dots & \lambda_2 \\ \lambda_2 & 2\mu_3 & \lambda_3 & \dots & \lambda_3 \\ \lambda_2 & \lambda_3 & 2\mu_4 & \dots & \lambda_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_2 & \lambda_3 & \lambda_4 & \dots & 2\mu_N \end{bmatrix} \tag{8}$$

must be positive semidefinite for all $N \in \mathbf{N}$.

The reverse does not hold. To understand this note that

$$\mathbb{E}[\pi_N^0] = \mathbb{E}\left[\vec{q}_{N,-1}^T[(\lambda_1 - \mu_1)1_{N-1\times N-1} - \frac{1}{2}\Lambda_N]\vec{q}_{N,-1}\right]. \tag{9}$$

We will show that price manipulation is possible here even when the condition in Proposition 8 is met. To this end, consider the following example: $N = 3$, Ω can be partitioned into Ω_1, Ω_2, and Ω_3, and λ_2 is discrete with $\lambda_2(\Omega_1) = \{4\}$, $\lambda_2(\Omega_2) = \{2\}$, and $\lambda_2(\Omega_3) = \{0\}$. In addition, the probabilities are $\mathbb{P}(\Omega_1) = \mathbb{P}(\Omega_2) = 1/4$, $\mathbb{P}(\Omega_3) = 1/2$, and $\widehat{\lambda}_1 = \hat{\mu}_1 = \hat{\mu}_2 = \hat{\mu}_3 = 1$. In this case, the trading strategy $q_1 = 0$, $q_2 = -q_3 = \lambda_2$, gives $\mathbb{E}[\pi_3^0] = 8$. Thus, prices can be profitably manipulated although the condition in Proposition 8 is satisfied.

A sufficient condition for the absence of price manipulation therefore has to be stronger than the condition in Proposition 8. One that derives immediately from (9) is $\mu_1 \geq \lambda_1 \geq 0$, $\mathbb{P} - a.e.$, and Λ_N being positive semidefinite, $\mathbb{P} - a.e.$, for all $N \geq 2$.

However, when the price-update and price-impact slopes are deterministic, the necessary condition stated in Proposition 8 is obviously also sufficient for the absence of price manipulation.

If the price-update and price-impact slopes are nonrandom, the absence of quasi-arbitrage can be easily characterized as well. Again consider the simple case $N = 3$. Then, (NoQA) is tantamount to $\sup_{\vec{q}_3 \in \mathfrak{P}_3} \mathbb{E}[\pi_3^0] < \infty$, or equivalently, $\inf_{\vec{q}_3 \in \mathfrak{P}_3} \mathbb{E}[\vec{q}_{3,-1}^T \Lambda_{3,-1} \vec{q}_{3,-1}] > -\infty$. The necessity of (NoQA) is trivial. To prove sufficiency, note that $\inf_{\vec{q}_3 \in \mathfrak{P}_3} \mathbb{E}[\vec{q}_{3,-1}^T \Lambda_{3,-1} \vec{q}_{3,-1}] = -\infty$ implies $\vec{q}_{3,-1}^T \Lambda_{3,-1} \vec{q}_{3,-1} < 0$. Hence, if the ray $\theta(-q_2 - q_3, q_2, q_3)$, as $\theta \to \infty$, is contained in $\mathfrak{P}_3$, then $\mathbb{E}[\pi_3^0] = O(\theta^2)$ and $\mathrm{SR}[\pi_3^0] \to \infty$. If the ray $\theta(-q_2 - q_3, q_2, q_3)$ is not contained in $\mathfrak{P}_3$, Appendix A shows that quasi-arbitrage is still feasible.

Of course, it is always true that a positive semidefinite $\Lambda_{3,-1}$ implies the absence of quasi-arbitrage, but the reverse may not be true. To see this, consider the values, $N = 3$, $\lambda_1 = \frac{1}{2}$, $\lambda_2 = 5$, $\mu_2 = 1$, and $\mu_3 = 6$. For this case, price manipulation is possible but there is no quasi-arbitrage because $\mathfrak{P}_3$ is bounded.

Hence, from the above and $\mu_1 = \lambda_1$ we have the following.

Proposition 9 *If the price-update and price impact functions are nonrandom, then*

i. the absence of price manipulation in $\mathcal{M}_1'(\mathcal{P}_1', \Pi_1')$ is equivalent to Λ_N being positive semidefinite for all $N \in \mathbf{N}$; and

ii. the absence of quasi-arbitrage in $\mathcal{M}_2'(\mathcal{P}_2', \Pi_2')$ is equivalent to $\inf_{\overrightarrow{q}_N \in \mathfrak{P}_N} \mathbb{E}[\overrightarrow{q}_{N,-1}^T \Lambda_N \overrightarrow{q}_{N,-1}] > -\infty$ for all $N \in \mathbf{N}$.

Proposition 9 contains two important facts. The first is that for any given price-update sequence there exists a price-impact sequence that preserves the absence of price manipulation and quasi-arbitrage. This is a consequence of the price update slopes and immediate price-impact slopes being different: the μ_n's only have to be chosen high enough. The second fact is that it gives a specific computational criterion for testing for the absence of price manipulation and quasi-arbitrage.

While all μ_n's must be nonnegative, the signs of the λ_n's are ambiguous. Negative λ_n's are in discord with the interpretation that purchases signal good news about the asset's value. But in a pure price manipulation/quasi-arbitrage framework, negativity makes perfect sense. The main mechanism that makes price manipulation successful is the positive relation between price update and trading volume. If the λ_n's are negative, this mechanism would not work any more. For example, a purchase that drives up the price today but moves down future prices would erode the trader's ability to make money from trading.

If there is no temporary price impact ($\{\lambda_n\}_{n=1}^N = \{\mu_n\}_{n=1}^N$) and liquidity increases over time sufficiently, then price manipulation is possible, because Λ_N is not positive semidefinite. Too high a rise in liquidity enables the trader to lock in expected profits from price manipulation: he begins pushing up the price by consecutive purchases until the market becomes more liquid. He then sells the shares he is holding and makes profits since, due to the more liquid market, he can do the selling at an average price higher than the average purchase price. This strategy constitutes a quasi-arbitrage if also

$\inf_{\overrightarrow{q}_N \in \mathfrak{P}_N} \mathbb{E}[\overrightarrow{q}^T_{N,-1} \Lambda_N \overrightarrow{q}_{N,-1}] = -\infty$.

One can find simple sufficient conditions for both *i* and *ii* in Proposition 9 to hold, when there is no temporary price impact. Since $det\ \Lambda_n > 0$ for all $2 \leq n \leq N$ implies a positive semidefinite Λ_N, and since $det\ \Lambda_n$ can be computed recursively by

$$det\ \Lambda_n = 2\lambda_n\ det\ \Lambda_{n-1} - \lambda^2_{n-1}\ det\ \Lambda_{n-2} \quad \text{for } n \geq 3, \tag{10}$$

with initial conditions $det\Lambda_2 = 2\lambda_2$ and $det\Lambda_3 = \lambda_2(4\lambda_3 - \lambda_2)$, the complexity of constructing price-update slopes which exclude price-manipulation and quasi-arbitrage is only of linear order.

For nondecreasing and recurrent price-update functions (a recurrent sequence is the infinite repetition of the same finite series of real numbers), Proposition 9 and equation (10) imply that *price-update functions with nondecreasing slopes necessarily make price manipulation and quasi-arbitrage infeasible, and recurrent price-update functions must be time-stationary to rule out price manipulation and quasi-arbitrage.*

Note that the set of price-update and price-impact functions which precludes quasi-arbitrage (price manipulation) forms a cone. That is, if $\{\lambda_n, \mu_n\}_{n=1}^N$ and $\{\lambda'_n, \mu'_n\}_{n=1}^N$ both rule out quasi-arbitrage (price manipulation), then so does their sum $\{\lambda_n+\lambda'_n, \mu_n+\mu'_n\}_{n=1}^N$ and the nonnegative multiple $\{\mathtt{B}\lambda_n, \mathtt{B}\mu_n\}_{n=1}^N$, $\mathtt{B} \geq 0$, which correspond to the price-update functions $\{\lambda_n+\lambda'_n\}_{n=1}^N$ and $\{\mathtt{B}\lambda_n\}_{n=1}^N$, and the price-impact functions $\{\mu_n + \mu'_n\}_{n=1}^N$ and $\{\mathtt{B}\mu_n\}_{n=1}^N$, respectively.

Note that the condition $\inf_{\overrightarrow{q}_N \in \mathfrak{P}_N} \mathbb{E}[\overrightarrow{q}^T_{N,-1} \Lambda_N \overrightarrow{q}_{N,-1}] > -\infty$ can be verified using standard constrained optimization software. Since, in case Λ_N is not positive semidefinite (only interesting case), the quadratic form descends from the origin in all directions in which $\inf_{\overrightarrow{q}_N \in \mathfrak{P}_N} \mathbb{E}[\overrightarrow{q}^T_{N,-1} \Lambda_N \overrightarrow{q}_{N,-1}] = -\infty$, one would always use 0 as the start value and check whether $\mathfrak{P}_N$ is ever left when descending. Note that typically $\inf_{\overrightarrow{q}_N \in \mathfrak{P}_N} \mathbb{E}[\overrightarrow{q}^T_{N,-1} \Lambda_N \overrightarrow{q}_{N,-1}] > -\infty$ if and only if Λ_N is positive semidefinite holds in the case where the price impact is only permanent and the price slopes are nonnegative. This is because quasi-arbitrage can be implemented by a sequence of purchases that pushes up the price, followed by a sequence of sales that moves down the price by less than the upwards pull of the original buys.

Proposition 7 regarding the relation between the absence of quasi-arbitrage and market viability is literally true for nonstationary price-update and price-impact functions, of course. So, there is no need to discuss this matter further. Only recall the insider and liquidity trading problem mentioned in the previous section. Those problems can also be studied in the present framework. As above, both problems are solvable if and only if $\mathcal{M}_2'(\mathcal{P}_2', \Pi_2')$ ($\mathcal{M}_1'(\mathcal{P}_1', \Pi_1')$) is viable. In the next subsection we will examine the insider trading problem based on the Kyle (1985) model in some more detail.

Note that nonlinear price-update functions may assume "chaotic" shapes without giving rise to quasi-arbitrage. For example, consider any strictly increasing nonnegative sequence $\{\lambda_n\}_{n=1}^{\infty}$, and suppose that a sequence of functions $\{U_n\}_{n=1}^{\infty}$ satisfies $U_1 = 0$ and $\lambda_{n-1}q \leq U_n(q) \leq \lambda_n q$ if $q \geq 0$ and $\lambda_n q \leq U_n(q) \leq \lambda_{n-1}q$ if $q < 0$, for all $n \geq 2$. Then, from (10) follows that this sequence of price-update functions does not allow quasi-arbitrage even though their shapes are quite arbitrary.

B Discussion of the Kyle Model

Black (1995) conjectures that the Kyle (1985) model allows "arbitrage opportunities" for uninformed agents if they pretend to be informed. We will prove that this conjecture is wrong for the monopolistic version of the Kyle model, but valid for the model with multiple insiders, given reasonable parameter values. We also demonstrate below that the equilibrium price-update functions in Kyle must be linear asymptotically.

Kyle's model describes a game between a competitive market-maker, who sets the price in each period, and an individual risk-neutral insider trader, who has information on the liquidation value $v > 0$ of the single asset that is traded. The framework is as follows. All trades take place in the time interval $[0, 1]$ and $\mathcal{D} = \mathcal{D}_M = \mathcal{D}_\eta = \mathcal{D}_\varepsilon = \mathbf{R}$. In each period, the market-maker observes only the aggregate trading volume, which is the sum of the insider's trading quantity and residual trades. He cannot observe the insider's amounts. Knowing the history of trades and the fact that there is one informed trader who maximizes his profits, he sets the price equal to the conditional expected value, that is, $p_n = \mathbb{E}_n[v]$ for $1 \le n \le N$.

The insider, on the other hand, taking into account how the market-maker computes the price, submits in each round the quantity that maximizes his profits. He solves $\sup_{\{q_n\}_{n=1}^N} \mathbb{E}[\sum_{n=1}^N (v - p_n) q_n]$, where dynamic consistency requires the optimal trading strategy $\{q_n\}_{n=1}^N$ to satisfy $\sup_{\{q_n\}_{n=j}^N} \mathbb{E}[\sum_{n=j}^N (v - p_n) q_n]$ for $1 \le j \le N$.

As Kyle shows for the case of normally distributed residual trades $\eta'_n \sim NID(0, \sigma^2)$, this game has a unique linear equilibrium where the price evolves according to $p_n = p_{n-1} + \lambda_n q_n + \eta'_n$, the liquidity parameters λ_n being endogenously (but deterministically) determined. But this price process is just a special case of (7) if the price impact has no temporary component, $\varepsilon_n = 0$, and $\eta'_n = \lambda_n \eta_n$ is set. This and the existence of the second moment of η'_n imply that the Kyle model can be categorized as a $\mathcal{M}'_2(\mathcal{P}'_2, \Pi'_2)$ market and thus all the results above can be applied to Kyle.

For small and large N, Kyle's slopes are almost constant and hence quasi-arbitrage is infeasible. Consequently, Kyle's equilibrium is viable. If the matrix Λ_N were not positive semidefinite, quasi-arbitrage would always be possible, because the equilibrium slopes are decreasing over time, which has $\inf_{\overrightarrow{q}_N \in \mathfrak{P}_N} \mathbb{E}[\overrightarrow{q}^T_{N,-1} \Lambda_N \overrightarrow{q}_{N,-1}] = -\infty$ as a consequence. For the Kyle model one has therefore an equivalence between the absence of quasi-arbitrage and Λ_N being positive semidefinite.

In contrast, the Kyle model with multiple (equally informed) insiders exhibits no viable equilibria if the number of insiders or auctions is sufficiently large. As Holden and Subrahmanyam (1992) prove, insiders trade very aggressively in early periods resulting in very low liquidity in the beginning. However, only after a few periods almost all insider information is impounded in the price and liquidity alters abruptly to higher levels. This change in liquidity occurs too fast in the sense that it violates the conditions that are put forward in Proposition 9.

What really happens is simply that dynamic programming can no longer be used to compute each insiders' problem $\sup_{\{q_n\}_{n=1}^N} \mathbb{E}[\sum_{n=1}^N (v - p_n) q_n]$, if Λ_N is not positive semidefinite. More precisely, the set of the latter problem's solutions does not coincide with the solution set of the Bellman equation

$$\pi_n = \sup_{q_n} \mathbb{E}_n[(v - p_n)q_n + \pi_{n+1}]. \tag{11}$$

A numerical example illustrates this point. Repeating the simulation shown in Figure 1 in Holden and Subrahmanyam (1992) for two insiders and twenty auctions, given the parameter values $Var[v] = \sigma^2 = 1$, we obtain the decreasing sequence $\{\lambda_n\}_{n=1}^{20}$ in their figure. Since $\lambda_2 = 1.69$ and $\lambda_7 = 0.4$, Λ_{20} is not positive semidefinite. Note that in a Nash equilibrium, in any period n, each insider takes the strategy of the other insider, $\hat{q}_{-n}$, as given. In particular, if $v > p_0$ ($v < p_0$), then $\hat{q}_{-n} > 0$ ($\hat{q}_{-n} < 0$) is expected. As a consequence, the strategy $q_1 = q_2 > 0$, $q_7 = -2q_1$, and $q_n = 0$ for all remaining n, renders $\mathbb{E}[\sum_{n=1}^{20}(v - p_n)q_n] = \mathbb{E}[\pi_{20}^0] \geq 0.09q_1^2 + 1.69q_1\hat{q}_{-2} + 0.8q_1\hat{q}_{-3} \to \infty$ as $q_1 \to \infty$. But, the Bellman equation (11) can be solved recursively for the same parameters; the results are depicted in Holden and Subrahmanyam (1992).

As for the monopolistic insider model, quasi-arbitrage is feasible due to increasing liquidity (all trades satisfy $q_n \in \mathfrak{P}_N$). Hence, "equilibria" in the multiple-insider Kyle model are not viable if one of the technical conditions in Proposition 9 is violated. It is therefore crucial to always check these conditions after equilibria have been constructed using dynamic programming.

One could try to extend the Kyle model by allowing prices that incorporate more general nonlinear price-update functions, like $p_n = p_{n-1} + U_{n,N}(q_n + \eta_n)$. We write here $U_{n,N}$ explicitly as functions of N, because Kyle fixes the number of auctions and we are interested in the case where N becomes large. Since the absence of quasi-arbitrage is necessary in equilibrium, it can be used to find the shape of equilibrium price-update functions. To simplify the analysis, we only consider price-update functions that are taken from the set $\mathfrak{F}$ of functions $\{U_{n,N}\}_{n=1}^{N}$ which are (i) smooth (twice continuously differentiable), (ii) symmetric, i.e., $U_{n,N}(x) = -U_{n,N}(-x)$, $x \geq 0$, (iii) monotone in the sense that $\hat{U}_{n,N}(x) \leq \hat{U}_{n-1,N}(x)$ for $x \geq 0$, and (iv) meet the conditions of market $\mathcal{M}_2'(\mathcal{P}_2', \Pi_2')$. The smoothness assumption is needed to calculate the insider's optimum; the second property says that the market maker treats purchases and sales symmetrically, and the third states that the insider's optimal policy causes the market maker to react less sensitively to trade size over time.

Proposition 10 *Suppose the price-update functions* $\{U_{n,N}\}_{n=1}^{N} \in \mathfrak{F}$ *constitute an equilibrium in the Kyle (1985) model when* N *auctions are performed. Then, the absence of quasi-arbitrage implies that the expected price-update functions converge pointwise to a linear function on any interval* $(\tau, 1-\tau)$, $0 < \tau < 1$, *as* $N \to \infty$. *More formally, for any* $\varepsilon > 0$ *there exists an index* N_0 *such that for* $N \geq N_0$

$$\left|\hat{U}_{n,N}(x) - \lambda x\right| < \varepsilon \quad n \in \mathbf{N} \text{ with } \tau \leq \frac{n}{N} \leq 1-\tau,\ x \in \mathbf{R},\ \lambda \geq 0.$$

Thus, as the number of auctions becomes very large, only equilibria with approximately linear expected price-update functions are viable. For a fixed number of auctions it is difficult to find the shapes of viable price-update functions. This is because the simultaneous computation of $\mathbb{E}[p_n]$ and $\sup_{\{q_n\}_{n=1}^{N}} \mathbb{E}[\sum_{n=1}^{N}(v-p_n)q_n]$ is complicated when all $U_{n,N}$'s are nonlinear.

VI Multiple Assets

So far we have discussed quasi-arbitrage only for one financial asset, but in typical applications investors trade many assets at the same time. In this section we extend our approach to the multivariate setting where a portfolio of $K > 1$ assets can be traded.

The multivariate case contains several interesting features not captured by the single-asset analysis. Presumably, the most important one is the ability to incorporate *cross-price impact.* If the traded quantity of asset i affects not only the price of asset i but also the prices of other assets, then price manipulation- and quasi-arbitrage opportunities are much richer.

Fortunately, all results proved for the single-asset case in Sections IV.A and V.A are literally true for the multi-asset case. Only recall that the slope λ in the definition of quasi-linearity (see Definition 3 above) now is a positive semidefinite matrix and not a nonnegative real number as in the single-asset case. In what follows, we outline the multi-asset versions of the proofs of Propositions 1 and 2, and then discuss the multi-asset proof of Proposition 5.

For this purpose we define $\hat{U}_{ij}(q) \in \mathbf{R}$ to be the expected price update of asset i when $q \in \mathcal{D}_M^j \subseteq \mathbf{R}$ shares of asset j and none of the other assets are traded. The outline of the proof below first discusses only the consequences of the absence of price manipulation and δ-arbitrage. Quasi-arbitrage is treated subsequently.

Step 1: $\hat{U}_{ij}$ is linear. As in the single-asset case, we prove this after we have shown that $\hat{U}_{ij}$ is symmetric on $\mathcal{D}_M^j$. Note that we have established already the linearity of $\hat{U}_{ii}$ in Proposition 1. Thus we only need to consider the case $i \neq j$ here.

Suppose $\hat{U}_{ij}(q) > -\hat{U}_{ij}(-q)$ for a $q > 0$, contradicting symmetry. Then the trading strategy of buying q shares of asset i in each of the first m periods, buying q shares of asset j in each of the next m periods, selling q shares of asset j in each of the following m periods, and selling q shares of asset i in each of the next m periods would yield $\mathbb{E}[\pi^0_{2m}] > 0$ and $\mathrm{SR}(\pi^0_{2m}) > \delta$ (if m and p_0 are sufficiently large). Similarly, $\hat{U}_{ij}(q) < -\hat{U}_{ij}(-q)$ for a $q > 0$ and $\hat{U}_{ij}(0) \neq 0$ can be rejected if either price manipulation or δ-arbitrage is ruled out.

Next, we argue that $\hat{U}_{ij}$ must be linear. Again, by way of contradiction, assume that there exists a $q \in \mathbf{Q}_+$ such that $\hat{U}_{ij}(q) > \hat{U}_{ij}(1)q$. Then buying q shares of asset i in each the first m periods, buying q shares of asset j in each of the next m periods, selling one share of asset j in each of the next mq periods, and selling q shares of asset i in each of the next m period gives $\mathbb{E}[\pi^0_{2m}] > 0$ and $\mathrm{SR}(\pi^0_{2m}) > \delta$. Analogously, $\hat{U}_{ij}(q) < \hat{U}_{ij}(1)q$ can be excluded.

Step 2: $\hat{U}_{ij} = \hat{U}_{ji}$, i.e., cross-price updates are symmetric. Suppose, on the contrary, $\hat{U}_{ij}(q) > \hat{U}_{ji}(q)$ for a $q > 0$. This says that trading asset j has a stronger impact on the price of asset i than the other way round. Then, the trading strategy of buying q shares of asset i in each of the first m periods, buying q shares of asset j in each of the next m periods, selling q shares of asset i in each of the next m periods, and selling q shares of asset j in each of the next m periods represents a price manipulation (and δ-arbitrage) if m is big enough. The mechanism of this strategy is clear: the expected gain from putting the purchase of asset j shares between the purchase and the sale of asset i shares outweighs the expected losses that derive from selling asset i shares between the purchase and the sale of asset j shares. Similarly, the inequality $\hat{U}_{ij}(q) < \hat{U}_{ji}(q)$ for a $q > 0$ cannot hold.

Step 3: $\hat{U}_i$ is additive separable, that is, $\hat{U}_i(q_1, q_2, \ldots, q_K) = \sum_{j=1}^{K} \hat{U}_{ij}(q_j)$. The trading strategies verifying the last equality are a bit more involved and relegated to Appendix A. Hence there exists a symmetric matrix λ such that $\hat{U}(q) = \lambda q$, for all $q \in \mathbf{R}$. The simplest way to see that λ has to be positive semidefinite is to assume $U = P$ and to calculate $\mathbb{E}[\pi_2^0] = -q^T \lambda q$ for $q_1 = -q_2 = q$. This expression has to be nonpositive for all $q \in \mathbf{R}^K$ to rule out price manipulation. The case $U \neq P$ is discussed in Appendix A. The remaining claims in (the multidimensional version of) Proposition 1 can be shown as in the single-asset case.

To find a necessary condition for the absence of quasi-arbitrage, we modify two assumptions of the market model $\mathcal{M}_2'(\mathcal{P}_2', \Pi_2')$. First, the assumption $\widehat{U}_n(q) \geq -\widehat{U}_n(-q)$ for all $q \in \mathcal{D}_M$, $q \geq 0$, is replaced by the postulate that all $\hat{U}_{ij}$ and $\hat{U}_i$ are symmetric. Second, $V(q; P, N) = O(N^{\gamma} 1_{K\times K})$, $V(q; U, N) = O(N^{\zeta} 1_{K\times K})$, and $Var[\varepsilon_N] = O(N^{\vartheta} 1_{K\times K})$, where $\gamma < 0$, $\zeta < 0$, and $\vartheta < 0$, or in words, the total variance in the time interval $[0, 1]$ cannot grow faster than linearly. If these adjustments are made, basically the same strategy as used above for the case of price manipulation and δ-arbitrage, can be applied to prove that U is necessarily quasi-linear if quasi-arbitrage is ruled out. The details of the arguments are explained in Appendix A. Hence, the multi-asset versions of Propositions 1 and 2 are proved.

Next, we demonstrate that the reverse of Propositions 1 and 2 is true as well if $U = (I_K - \alpha)P$. To this end, we use the same approach as in the single-asset case to find a sufficient condition for the absence of quasi-arbitrage. In particular, if both the price-update and price-impact functions are quasi-linear, then the multi-asset version of Lemma 1 holds when the condition $P(x) \geq \frac{1}{2}U(x)$ for $x \geq 0$, $x \in \mathcal{D}$, is replaced by the postulate that $P - \frac{1}{2}U$ is positive semidefinite (For a detailed analysis of multidimensional optimal trading problems of this kind see Huberman and Stanzl (2001).). Therefore we conclude that Proposition 5 is valid also for multiple assets (note that $P - \frac{1}{2}U = \frac{1}{2}(1_K + \alpha)P$ is positive semidefinite).

One important consequence is that, in absence of a temporary price impact, a multi-asset environment with nonzero cross-price effects can always be reduced to one that exhibits *no* cross-price impact. To understand this, note that any positive semidefinite matrix λ can be written as the product $C^T \Psi C$ of a diagonal matrix Ψ, which diagonal is formed by the nonnegative eigenvalues of λ, and a matrix C constructed by the eigenvectors of λ. If we interpret the entries of C as portfolio weights of the underlying assets, then C is a collection of K portfolios. If we replace the original assets with these portfolios, the relevant price-update function becomes $U_C(q) = \Psi q$, which incorporates no cross-price impact.

Black (1995) informally argues that the sum of the price update of individual trades must equal the price update of trading the "basket" containing these individual trades. In other words, the price update must be an additive function in the trading volume. Our results demonstrate that eliminating quasi-arbitrage requires more structure on the shape of the price-update function than Black claims.

Multi-asset versions of Propositions 8 and 9 can also be formulated. However, we refrain from stating them since they do not provide any further qualitative insights, but only add considerably more notation.

VII Price Manipulation and the Gain-Loss Ratio

Bernardo and Ledoit (2000) propose to use the "gain-loss" ratio of an investment as a measure of its attractiveness. It is defined as the expectation of the investment's positive excess payoffs divided by the expectation of its negative excess payoffs. More formally, if z denotes the payoff of a zero-cost portfolio, then the gain-loss ratio equals $\mathrm{GLR}[z] \triangleq \mathbb{E}[z^+]/\mathbb{E}[z^-]$, where $z^+ = \max(z, 0)$ and $z^- = \max(-z, 0)$.

In the framework of Bernardo and Ledoit (2000) the absence of pure arbitrage is equivalent to the gain-loss ratio being finite. Thus, by imposing an upper bound on the gain-loss ratio arbitrage opportunities are ruled out. Even though our model generally does not exhibit this equivalence, one might want to exclude zero-cost investment opportunities, π_N^0, with $\lim_{N\to\infty} \mathbb{E}[\pi_N^0] = \lim_{N\to\infty} \mathrm{GLR}[\pi_N^0] = \infty$. The existence of such a "good deal", let us call it a *quasi*-arbitrage*, may threaten market viability.

One advantage of using the gain-loss ratio to detect attractive investment opportunities, rather than the Sharpe ratio, is that it recognizes a pure arbitrage with fat upper tails and flat lower tails as desirable, while the Sharpe ratio does not. However, in our setup the gain-loss-ratio approach has the disadvantage that there is no simple utility-based definition of market viability which could give rise to an equivalence between the absence of quasi*-arbitrage and market viability.

Now, what is the relation between the absence of quasi*-arbitrage and the shape of the price-update and price-impact functions? To answer this question we first need to specify the price model. Call it $\mathcal{M}_1''(\mathcal{P}_1'', \Pi_1'')$. Its properties are $\mathcal{M}_1''(\mathcal{P}_1'', \Pi_1'') \subseteq \mathcal{M}_1'(\mathcal{P}_1', \Pi_1')$, $\mathbb{E}[P_n(q + \eta_N)^-] = O(N^\gamma 1_{K\times K})$, $\mathbb{E}[U_n(q+\eta_N)^-] = O(N^\zeta 1_{K\times K})$, and $\mathbb{E}[\varepsilon_N^-] = O(N^\vartheta 1_{K\times K})$ for all $q \in \mathcal{D}_M$, $q > 0$, where $\gamma < 1$, $\zeta < 0$, and $\vartheta < 0$. These assumptions resemble those of market $\mathcal{M}_2'(\mathcal{P}_2', \Pi_2')$, except that they are defined on the negative part of the random variables rather than on their variances.

Under these conditions we obtain similar results to Propositions 1, 2, 5, 9, and 10. We find it convenient to list them all together.

If the price update and price impact are time-stationary:

- *Proposition 1**: The absence of quasi*-arbitrage in $\mathcal{M}_1''(\mathcal{P}_1'', \Pi_1'')$ implies that U is quasi-linear.
- *Proposition 2**: Under the distributional assumptions in Proposition 2, the price-update function in Proposition 1* is exactly linear.
- *Proposition 5**: If $U = (1-\alpha)P$, then the absence of quasi*-arbitrage in $\mathcal{M}_1''(\mathcal{P}_1'', \Pi_1'')$ is equivalent to the quasi-linearity of U.

If the price update and price impact are nonstationary:

- *Proposition 9**: Suppose that the price-update and price-impact functions are nonrandom. Then, the absence of quasi*-arbitrage in $\mathcal{M}_1''(\mathcal{P}_1'', \Pi_1'')$ is equivalent to $\inf_{\vec{q}_N \in \mathfrak{P}_N} \mathbb{E}[\vec{q}_{N,-1}^{\,T} \Lambda_N \vec{q}_{N,-1}] > -\infty$ for all $N \in \mathbf{N}$.

- *Proposition 10**: Under the assumptions in Proposition 10, the absence of quasi*-arbitrage causes the equilibrium price-update functions in the Kyle (1985) model to converge in the same way as it is required by the absence of quasi-arbitrage.

The multi-asset versions of Propositions 1*, 2*, 5*, and 9* can also be demonstrated to hold. Summarizing, postulating the absence of quasi*-arbitrage in $\mathcal{M}_1''(\mathcal{P}_1'', \Pi_1'')$ is equivalent to imposing the absence of quasi-arbitrage in $\mathcal{M}_2'(\mathcal{P}_2', \Pi_2')$. Both conditions are linked through the quasi-linearity of the price-update and price-impact functions.

VIII Concluding Remarks

This paper introduces the concept of quasi-arbitrage for markets where trade size moves the price and prices are uncertain when trades are placed. A quasi-arbitrage is a zero-cost trading strategy that creates an infinite expected payoff, as well as an infinite Sharpe ratio of the payoff. Markets are viable if and only if there is no quasi-arbitrage, when agents' utility is measured by the mean and standard deviation of an investment opportunity and agents are not too risk averse.

We examine the conditions imposed by the absence of quasi-arbitrage on the functional shape of the temporary and permanent price effect of a trade. If the price-update and price-impact functions are stationary and multiples of each other, then the absence of quasi-arbitrage is equivalent to the linearity of both functions. On the other hand, if the price-update and price-impact functions are independent, then only the price-update function must be linear in trading volume, while the temporary price impact can have various forms without offering quasi-arbitrage opportunities.

The theoretical micro-structure literature usually assumes that the change in prices is time-independent and reacts linearly to trading volume. This paper demonstrates that the assumption of stationarity of price changes already implies the linearity of the price-update function.

Linearity as a necessary condition for the absence of quasi-arbitrage calls for a careful examination of empirical estimations of price-update functions. To the extent that they detect deviations from linearity, one can suspect some misspecification (perhaps a nonstationary environment) or wonder if indeed some arbitrage possibilities had gone unexploited.

Postulating a finite gain-loss ratio instead of the absence of quasi-arbitrage does not change any of our conclusions. Also in this case the price-update function has to be linear, since otherwise the gain-loss ratio would become infinite.

The results of this paper ask for one main extension, namely to permit the trading of market and limit orders at the same time. How do limit orders affect the market price? And what does a no-arbitrage condition look like if traders can submit market and limit orders simultaneously? Most important, we would like to examine how market and limit orders can coexist in an equilibrium exchange.

Appendix A. Proofs of the results in Sections IV-VI

Before proving Propositions 1 and 2, we derive two very useful results. To simplify the analysis, we assume throughout this paper that if $q \in \mathcal{D}_M$ has an irrational component, then there exists at least one sequence of vectors $\{q_j^{(q)}\}_{n=1}^{\infty}$ such that all $q_n^{(q)} \in \mathcal{D}_M \cap \mathbf{Q}^K$ and $\lim_{n\to\infty} q_n^{(q)} = q$.

Lemma 2 *Each of the conditions (NoQA)-(NoM) implies:*

i. $\hat{U}$ *is symmetric on* $\mathcal{D}_M$*, i.e.,* $\hat{U}(q) = -\hat{U}(-q)$ *for* $q \in \mathcal{D}_M$*; and*

ii. $\lim_{n\to\infty} \hat{U}(q_n^{(q)}) = \hat{U}(q)$ *when* $q \in \mathcal{D}_M \backslash \{0\}$ *is irrational and* $\lim_{n\to\infty} q_n^{(q)} = q$*, all* $q_n^{(q)} \in \mathbf{Q}$.

Proof. To verify i we start by proving that $\hat{U}(q) \leq -\hat{U}(-q)$ holds for $q \in \mathcal{D}_M \cap \mathbf{R}_+$. Suppose that this is not true, that is, there exists a $q > 0$ with $\hat{U}(q) > -\hat{U}(-q)$. Implement now the following trading strategy: buy in each of the first m periods the volume q, and then sell the quantity q in each of the next m periods. The expected profit of this round-trip strategy is

$$\mathbb{E}[\pi_{2m}^0] = \mathbb{E}[-\sum_{n=1}^{2m} p_n q_n] = \frac{m^2}{2} q[\hat{U}(q) + \hat{U}(-q)] - \frac{m}{2} q[\hat{U}(-q) - \hat{U}(q) + 2(\hat{P}(q) - \hat{P}(-q))] + c(2m)$$

(note that $N \geq 2m$ by assumption). If the variances exist, we can calculate

$$Var[\pi_{2m}^0] = \sum_{n=1}^{m} Var[(m-n+1)U(q+\eta_n) - P(q+\eta_n)]$$

$$+ \sum_{n=1}^{m-1} Var[(m-n)U(-q+\eta_{m+n}) + P(-q+\eta_{m+n})] + V(-q;P,N) + \frac{m(2m^2-1)}{3} Var[\varepsilon_N]$$

$$\leq \frac{(2m+1)(m+1)m}{6} V(q;U,N) + mV(q;P,N) + m(m+1)\sqrt{V(q;U,N)V(q;P,N})$$

$$+\frac{(2m-1)m(m-1)}{6} V(-q;U,N) + (m-1)V(-q;P,N) + m(m-1)\sqrt{V(-q;U,N)V(-q;P,N})$$

$$+V(-q;P,N) + \frac{m(2m^2-1)}{3} Var[\varepsilon_N]$$

thanks to Minkowski's inequality. Hence, $\mathbb{E}[\pi^0_{2m}] = O(m^2)$ and $Std[\pi^0_{2m}] = o(m^\theta)$, $\theta < 2$, which contradicts each of (NoQA)-(NoM).

Next, we show $\hat{U}(q) \geq -\hat{U}(-q)$ for $q \in \mathcal{D}_M \cap \mathbf{R}_+$, also by contradiction (note that only the cases (NoM) and (NoδA) need to be treated). For this purpose assume a $q > 0$ satisfying $\hat{U}(q) < -\hat{U}(-q)$. Now, selling in each of the first m periods the quantity q and then buying the volume q in each of the following m periods results in $\mathbb{E}[\pi^0_{2m}] = O(m^2)$ and $Std[\pi^0_{2m}] = o(m^\theta)$, $\theta < 2$, where m is such that $\mathbb{E}[p_n] \geq 0$ for all $1 \leq n \leq 2m$ (note that m must be finite). Thus, given a sufficiently large initial price, (NoQA)-(NoM) are all violated.

The second assertion, *ii*, is easiest shown by contradiction, too. Assume that $q \in \mathcal{D}_M \backslash \{0\}$ is irrational, that $\{q^{(q)}_n\}^\infty_{n=1}$ is a sequence of rational numbers such that $\lim_{n\to\infty} q^{(q)}_n = q$, and that *ii* does not hold, i.e., there exists a $q > 0$ (we can choose a positive q due to *i*) and $\varepsilon > 0$ such that one the following cases applies:

1. there exists a subsequence $q^{(q)}_{n'} \to q+$ with $\hat{U}\left(q^{(q)}_{n'}\right) \geq \hat{U}(q) + \varepsilon$,
2. there exists a subsequence $q^{(q)}_{n'} \to q+$ with $\hat{U}\left(q^{(q)}_{n'}\right) \leq \hat{U}(q) - \varepsilon$,
3. there exists a subsequence $q^{(q)}_{n'} \to q-$ with $\hat{U}\left(q^{(q)}_{n'}\right) \geq \hat{U}(q) + \varepsilon$,
4. there exists a subsequence $q^{(q)}_{n'} \to q-$ with $\hat{U}\left(q^{(q)}_{n'}\right) \leq \hat{U}(q) - \varepsilon$.

We shall show that U violates (NoQA)-(NoM) in each case.

Case 1. Use the following strategy: buy $q^{(q)}_{n'}$ units of the asset in each of the first m periods, where n' is an arbitrary index of the subsequence; then sell the quantity q in the each of the following m periods. Given *i*, the mean and volatility of these transactions' profit are $\mathbb{E}[\pi^0_{2m}] = O\left(\frac{m^2}{2}[(\hat{U}(q^{(q)}_{n'}) - \hat{U}(q))q + \hat{U}(q^{(q)}_{n'})(q - q^{(q)}_{n'})]\right)$ and $Std[\pi^0_{2m}] = o(m^\theta)$, $\theta < 2$. Since the coefficient in the former term is positive for sufficiently large n' (verify that the sequence $\left\{\hat{U}\left(q^{(q)}_{n'}\right)\right\}$ must be bounded!), a contradiction to each of (NoQA)-(NoM) is established.

Case 2. Trading strategy: buy volume q in the each of the first m periods and then sell $q_{n'}^{(q)}$ units in each of the next $m-1$ periods. This implies $\mathbb{E}[\pi_{2m-1}^{0}] = O\left(\frac{m^2}{2}[(\hat{U}(q) - \hat{U}(q_{n'}^{(q)}))q_{n'} - \hat{U}(q)(q_{n'}^{(q)} - q)]\right)$ and and $Std[\pi_{2m-1}^{0}] = o(m^{\theta})$, $\theta < 2$. But the coefficient in the first expression becomes positive if n' is sufficiently large (note that $(m-1)\, q_{n'}^{(q)} \leq mq$ is met if n' is large enough). Again, (NoQA)-(NoM) are all invalid.

The reader can easily check that for the remaining cases the following two trading strategies contradict each of (NoQA)-(NoM): for case 3, buy $q_{n'}^{(q)}$ units in each of the first m periods and then sell quantity q in each of the following $m-1$ periods; for case 4, buy q units in each of the first m periods and then sell the volume $q_{n'}^{(q)}$ in each of the next m periods. □

Lemma 3 *Each of (NoQA)-(NoM) requires that U satisfies the linear integral equation*

$$\int_{\Omega} U(q+\eta)d\mathbb{P} = \lambda q \qquad \text{for all } q \in \mathcal{D}_M, \tag{12}$$

where η assumes the residual trades' distribution, and $\lambda \geq 0$.

Proof. Note that (12) is equivalent to $\hat{U}(q) = \lambda q$ for all $q \in \mathcal{D}_M$. To prove Lemma 3, suppose that $\hat{U}$ does not have the above property, i.e., there exists a $q > 0$, such that $\hat{U}(q) > \hat{U}(1)\, q$ or $\hat{U}(q) < \hat{U}(1)\, q$. Let us deal with the first case. Thanks to Lemma 2 *ii* we can choose q to be a rational number. Implement now the following trading strategy: buy q units of the asset in each of the first m periods such that mq is an integer, then sell one unit in each of the following mq periods. It follows that $\mathbb{E}[\pi_{m(1+q)}^{0}] = O(\frac{m^2}{2}q[\hat{U}(q) - \hat{U}(1)q])$ and $Std[\pi_{m(1+q)}^{0}] = o(m^{\theta})$, $\theta < 2$, contradicting each of (NoQA)-(NoM).

The case $\hat{U}(q) < \hat{U}(1)\, q$ can be tackled similarly: it is easy to verify that the strategy of buying one unit in each of the first mq periods and then selling q units in each of the next m periods results in a violation of each of (NoQA)-(NoM). This completes the proof. □

Proof of Proposition 1. From Lemma 3 we know that there exists a $\lambda \geq 0$ such that $\int_{\Omega} U(q+\eta)d\mathbb{P} =$

λq for all $q \in \mathcal{D}_M$, provided that either (NoQA), (NoUM), (NoδA), or (NoM) is valid. Take this λ and define the supplementary function S_U on $\mathcal{D}$ by $S_U(y) \triangleq U(y) - \lambda y$. The integral equation (12) can now be restated as

$$\int_\Omega S_U(q+\eta)d\mathbb{P} = 0 \quad \text{for all } q \in \mathcal{D}_M. \tag{13}$$

Having (13) at hand, we are ready to show that S_U satisfies the integral equation, $\mathbb{E}_n[S_U(\tilde{q}_n + \eta_n)] = 0$, for any $\mathcal{F}_n$-measurable random variable $\tilde{q}_n$. First, note that we are allowed to write (we employ $\mathcal{F}_n$ to denote both the vector of variables known at time n and the sigma-algebra it generates)

$$\mathbb{E}[S_U(\tilde{q}_n + \eta_n) \mid \mathcal{F}_n] = \mathbb{E}[S_U(g(\mathcal{F}_n) + \eta_n) \mid \mathcal{F}_n = .] \circ \mathcal{F}_n,$$

since, due to the Doob-Dynkin lemma, there exists a Borel measurable function $g : \mathbf{R}^{4n-2} \to \mathcal{D}$ such that $\tilde{q}_n = g(\mathcal{F}_n)$. Then, using the notation $\mathbb{P}_{\eta_n|\mathcal{F}_n=x}$ for the distribution of η_n given the event $\{\mathcal{F}_n = x\}$, we obtain

$$\mathbb{E}[S_U(g(\mathcal{F}_n) + \eta_n) \mid \mathcal{F}_n = x] = \int_{\mathbf{R}} S_U(g(x)+y)d\mathbb{P}_{\eta_n|\mathcal{F}_n=x}(y)$$

$$= \int_{\mathbf{R}} S_U(g(x)+y)d\mathbb{P}_{\eta_n}(y) = 0$$

for all $x \in \mathbf{R}^{4n-2}$, thanks to (13) and the independence of η_n. Therefore, $\mathbb{E}[S_U(\tilde{q}_n + \eta_n) \mid \mathcal{F}_n] = 0$, and U is quasi-linear. $\square$

Proof of Proposition 2. We only have to study here equation (13).

If $\mathbb{P}[\eta_n = 0] = 1$, then $U(q) = \lambda q$, $L(\mathbf{R}) - a.e.$, follows immediately from (13).

To simplify the analysis for case *ii*, we assume that there exists a number $a \in (0,1)$ (preferably close to one) such that the function $x \longmapsto U(x)e^{-a\frac{x^2}{2\sigma_\eta^2}}$ is $L(\mathbf{R})$-integrable, where $\sigma_\eta^2 = Var[\eta_n]$. This is a mild assumption because $\mathbb{E}[U(\eta_n)] < \infty$ holds in market $\mathcal{M}_1(\mathcal{P}_1, \Pi_1)$.

For normally distributed η_n's the integral equation (13) becomes

$$\frac{1}{\sqrt{2\pi}\sigma_\eta}\int_{\mathbf{R}} S_U(x)e^{-\frac{(x-q)^2}{2\sigma_\eta^2}}\,dx = 0 \quad \text{for all } q \in \mathbf{R},\ \lambda \geq 0. \tag{14}$$

Using the above assumption, it is an easy exercise to verify that (14) can be reformulated as

$$\int_{\mathbf{R}} \left[S_U(x)e^{-a\frac{x^2}{2\sigma_\eta^2}}\right]\left[\frac{1}{\sqrt{2\pi}\sigma_\eta/\sqrt{1-a}}e^{-\frac{(x-q)^2}{2\sigma_\eta^2/(1-a)}}\right]dx = 0 \tag{15}$$

for all $q \in \mathbf{R}$, $\lambda \geq 0$.

Now, recall that the Fourier transform $F[f] : \mathbf{R} \to \mathbf{C}$ of a $L(\mathbf{R})$-integrable function $f : \mathbf{R} \to \mathbf{R}$ is defined by $F[f](x) \triangleq \int_{\mathbf{R}} e^{ixy} f(y)dy$. Invoking the convolution theorem of Fourier transforms for (15) gives

$$F\left[y \mapsto S_U(y)e^{-a\frac{y^2}{2\sigma_\eta^2}}\right](x)\ e^{-\frac{x^2}{2\sigma_\eta^2/(1-a)}} = 0 \quad \text{for all } x \in \mathbf{R},$$

which implies that $S_U = 0$, $L(\mathbf{R}) - a.e.$, since F is injective. So $U(q) = \lambda q$, $L(\mathbf{R}) - a.e.$, holds also for the case of normal-distributed η_n's. □

Remark 1 *The supplementary function of the price-update function in Proposition 1 is also zero when the residual trades are a transform $W : \mathbf{R} \to \mathbf{R}$ of a zero-mean normal random variable, where W satisfies $W(x) = -W(-x)$, $\frac{dW}{dx}(x) > 0$ for all $x \geq 0$, and $\lim_{x\to\infty} W(x) = \infty$.*

Proof. In this case, (13) has the form

$$\frac{1}{\sqrt{2\pi}\sigma_\eta}\int_{\mathbf{R}} S_U(q + W(x))e^{-\frac{x^2}{2\sigma_\eta^2}}\,dx = 0 \quad \text{for all } q \in \mathbf{R}. \tag{16}$$

But this is evidently equivalent to

$$\frac{1}{\sqrt{2\pi}\sigma_\eta}\int_{\mathbf{R}} \frac{S_U(x)}{W'(x)} \exp\left[-\frac{1}{2\sigma_\eta^2}\left(W^{-1}\right)^2(x-q)\right]dx = 0 \quad \text{for all } q \in \mathbf{R}. \tag{17}$$

By applying Fourier transforms to this equation we obtain

$$F\left[y \mapsto \frac{S_U(y)}{W'(y)}\right](x)\ F\left[y \mapsto \exp\left[-\frac{1}{2\sigma_\eta^2}\left(W^{-1}\right)^2(y)\right]\right](x) = 0$$

for all $x \in \mathbf{R}$, from which $S_U = 0$, $L(\mathbf{R}) - a.e.$, follows, because W' and the modulus of $F[y \mapsto \exp\left[-\frac{1}{2\sigma_\eta^2}\left(W^{-1}\right)^2(y)\right]]$ are both positive on $\mathbf{R}$. □

Proof of Lemma 1. Observe that $\sup\{\mathbb{E}[\pi_N^0] \mid \pi_N^0 \in \Pi\} = 0$ holds if the cost-minimizing problem

$$\inf_{\{q_n,\ \mathcal{F}_n-measurable\}_{n=1}^N} \mathbb{E}[\sum_{n=1}^N p_n q_n] \quad \text{subject to} \quad \sum_{n=1}^N q_n = 0 \tag{18}$$

has zero expected costs as its optimum, for any $N \in \mathbf{N}$. To solve (18), only the case $P > \frac{1}{2}U$ has to be considered since $\mathbb{E}[\pi_N^0] = 0$ when $P = \frac{1}{2}U$. It is convenient here to modify (18) slightly by replacing the constraint $\sum_{n=1}^N q_n = 0$ with $\sum_{n=1}^N q_n = Q \geq 0$. The associated Bellman equation of (18) with the more general constraint is

$$C_n = \min_{q_n,\ \mathcal{F}_n\text{-measurable}} \mathbb{E}_n[p_n q_n + C_{n+1}] \tag{19}$$

$$\text{subject to} \quad Q_n = Q_{n-1} - q_{n-1},$$

$$Q_0 \triangleq 0,\ Q_1 \triangleq Q \geq 0,\ \text{and } Q_{N+1} = 0,$$

where the Q_n's denote the remaining shares to be traded, and C_n represents the remaining expected costs of trading. Standard computations show that optimal trades and cost function have the form

$$q_n = \frac{Q}{N} \geq 0, \quad \text{for } 1 \leq n \leq N, \text{ and}$$

$$\mathbb{E}[C_1] = p_0 Q + \frac{N+1}{2N}[2P(1) - U(1)]Q^2 \geq 0,$$

implying that (NoQA)-(NoM) are all valid. □

Proof of Proposition 9. The only technical difficulty here is to show that the absence of quasi-arbitrage implies $\inf_{\overrightarrow{q}_N \in \mathfrak{P}_N} \mathbb{E}[\overrightarrow{q}^T_{N,-1} \Lambda_N \overrightarrow{q}_{N,-1}] > -\infty$. Since this implication is trivial when $\mathfrak{P}_N$ is bounded, we can assume that $\mathfrak{P}_N$ is unbounded. If $\inf_{\overrightarrow{q}_N \in \mathfrak{P}_N} \mathbb{E}[\overrightarrow{q}^T_{N,-1} \Lambda_N \overrightarrow{q}_{N,-1}] = -\infty$, there exists a vector $x \in \mathfrak{I}$ taken from the nonempty set $\mathfrak{I} \triangleq \{y \in \mathbf{R}^{N-1} \mid y^T \Lambda_N y < 0\}$. Now, construct the half line $\overrightarrow{x} \triangleq \{x^\theta \mid x^\theta = \theta x, \theta \geq 0\}$. If there exists only one vector $x \in \mathfrak{I}$ which half line is contained in $\mathfrak{P}_N$, then $\mathbb{E}[\pi^0_N] = -\mathbb{E}[\frac{1}{2}(x^\theta)^T \Lambda_N x^\theta] = -\frac{1}{2}\theta^2 x^T \Lambda_N x \to \infty$ as $\theta \to \infty$, and $\mathrm{SR}[\pi^0_N] = O(\theta)$. But this contradicts the absence of quasi-arbitrage.

For the case where no $x \in \mathfrak{I}$ exist such that $\overrightarrow{x} \subset \mathfrak{P}_N$, introduce the ball $\mathfrak{B}_\rho \triangleq \{y \in \mathbf{R}^{N-1} \mid |y| \leq \rho\}$ and its boundary $\partial\mathfrak{B}_\rho \triangleq \{y \in \mathbf{R}^{N-1} \mid |y| = \rho\}$, where $\rho > 0$ is such that $\mathfrak{B}_\rho \subset \mathfrak{P}_N$. Furthermore, let us standardize the set $\mathfrak{I}$ by setting $\mathfrak{I}_\rho \triangleq \mathfrak{I} \cap \partial\mathfrak{B}_\rho$, which is motivated by the fact that $x \in \mathfrak{I}$ implies $x^\theta \in \mathfrak{I}$ and $-x^\theta \in \mathfrak{I}$, for all $\theta \geq 0$. If $\inf_{\overrightarrow{q}_N \in \mathfrak{P}_N} \mathbb{E}[\overrightarrow{q}^T_{N,-1} \Lambda_N \overrightarrow{q}_{N,-1}] = -\infty$, then there exists a sequence of vectors $\{x_n\}_{n=1}^\infty$, $x_n \in \mathfrak{I}_\rho$, such that $\lim_{n\to\infty} \mathbb{E}[(x_n^{\theta_n})^T \Lambda_N x_n^{\theta_n}] = -\infty$, $(-\sum_{j=1}^{N-1}(x_n^{\theta_n})_j, x_n^{\theta_n}) \in \mathfrak{P}_N$. In case this sequence satisfies $\mathbb{E}[(x_n^T \Lambda_N x_n] < -\varkappa$, $\varkappa > 0$, for all n, we have

$$\frac{-\mathbb{E}[(x_n^{\theta_n})^T \Lambda_N x_n^{\theta_n}]}{Std[(x_n^{\theta_n})^T \Lambda_N x_n^{\theta_n}]} \geq \frac{-\theta_n^2 \mathbb{E}[x_n{}^T \Lambda_N x_n]}{\theta_n Std_\rho^{\max}} \geq \frac{\varkappa}{Std_\rho^{\max}} \theta_n,$$

where $Std_\rho^{\max} \triangleq \max\{Std[y^T \Lambda_N y] \mid y \in \partial\mathfrak{B}_\rho\}$. Therefore, $\lim_{n\to\infty} \mathbb{E}[(\pi^0_N)_n] = \infty$ and $\lim_{n\to\infty} \mathrm{SR}[(\pi^0_N)_n] = \infty$ for $(\pi^0_N)_n \triangleq -\frac{1}{2}(x_n^{\theta_n})^T \Lambda_N x_n^{\theta_n}$, being at variance with the absence of quasi-arbitrage.

In contrast, if the above sequence does not meet $\mathbb{E}[(x_n^T \Lambda_N x_n] < -\varkappa$, $\varkappa > 0$, for all n , then there exists a subsequence $\{x_n\}_{n=1}^\infty$, $x_n \in \mathfrak{I}_\rho$, with $x_n \to x \in \partial\mathfrak{B}_\rho$, $\overrightarrow{x} \subset \mathfrak{P}_N$, and $\mathbb{E}[(x_n^T \Lambda_N x_n] \to 0$. We can assume that the corresponding vectors $x_n^{\theta_n}$ are selected such that $(-\sum_{j=1}^{N-1}(x_n^{\theta_n})_j, x_n^{\theta_n}) \in \partial\mathfrak{P}_N$, where $\partial\mathfrak{P}_N$ is the boundary of the polyhedral convex set $\mathfrak{P}_N$. Since the convergence of $\mathbb{E}[(x_n^T \Lambda_N x_n] \to 0$ if $x_n \to x$ takes place quadratically in $|x_n - x|$, but θ_n grows exponentially with $|x_n - x|$, we conclude that $\theta_n \mathbb{E}[(x_n^T \Lambda_N x_n] \to -\infty$. Hence,

$$\mathrm{SR}[(\pi^0_N)_n] = \frac{-\mathbb{E}[(x_n^{\theta_n})^T \Lambda_N x_n^{\theta_n}]}{Std[(x_n^{\theta_n})^T \Lambda_N x_n^{\theta_n}]} \geq \frac{-\theta_n \mathbb{E}[x_n{}^T \Lambda_N x_n]}{Std_\rho^{\max}} \to \infty$$

as $n \to \infty$. Again, this contradicts the absence of quasi-arbitrage. $\square$

Proof of Proposition 10. Take any $\varepsilon > 0$ and define $n(\tau, N) \triangleq \max\{j \in \mathbf{N} \mid \frac{j}{N} \leq \tau\}$. First, we demonstrate that for an arbitrary $q \geq 0$, $\hat{U}_{n(\tau,N),N}(q) \leq \hat{U}_{N-n(\tau,N),N}(q) + \varepsilon$, if N is sufficiently large. Suppose not, i.e., there exists a sequence $N_m \to \infty$ such that $\hat{U}_{n(\tau,N_m),N_m}(q) > \hat{U}_{N_m-n(\tau,N_m),N_m}(q) + \varepsilon$. Then buying q shares in each of the first $n(\tau, N_m)$ periods, and then selling q shares in each of the periods $N_m - n(\tau, N_m) + 1$, $N_m - n(\tau, N_m) + 2, \ldots, N_m$, results in $\mathbb{E}[\pi^0_{2n(\tau,N_m)}] = O(n(\tau, N_m)^2\varepsilon)$ and $Std[\pi^0_{2n(\tau,N_m)}] = O(n(\tau, N_m)\varepsilon)$. As $m \to \infty$, this becomes a quasi-arbitrage.

Next, we verify that the inequalities

$$q\hat{U}_{N-n(\tau,N),N}(1) - \varepsilon \leq \hat{U}_{N-n(\tau,N),N}(q) \leq q\hat{U}_{N-n(\tau,N),N}(1) + \varepsilon, \tag{20}$$

$q \geq 0$, must hold for sufficiently large N. If the second inequality were untrue, then there would exist a $q > 0$ and a sequence $N_m \to \infty$ such that $\hat{U}_{N_m-n(\tau,N_m),N_m}(q) > \hat{U}_{N_m-n(\tau,N_m),N_m}(1) + \varepsilon$. Note that the last paragraph guarantees the existence of an index m_0 such that $\hat{U}_{n,N_m}(1) \leq \hat{U}_{N_m-n(\tau,N_m),N_m}(1) + \frac{\varepsilon}{3q}$ for $m \geq m_0$ and $n(\tau, N_m) \leq n \leq N_m - n(\tau, N_m)$. For these indices we thus have $q\hat{U}_{n,N_m}(1) \leq q\hat{U}_{N_m-n(\tau,N_m),N_m}(1) + \frac{\varepsilon}{3}$ and $\hat{U}_{n,N_m}(q) \geq q\hat{U}_{n,N_m}(1) + \frac{\varepsilon}{3}$ as a consequence. In view of this, the trading strategy of buying q shares in each of the periods $n(\tau, N_m) + 1$, $n(\tau, N_m) + 2, \ldots, n(\tau, N_m) + n'_m$, where $n'_m \triangleq \max\{j \in \mathbf{N} \mid N_m - n(\tau, N_m) - j \geq qj \in \mathbf{N}\}$ (q can be chosen to be rational), and then selling 1 share in each of the periods $n(\tau, N_m) + n'_m + 1$, $n(\tau, N_m) + n'_m + 2, \ldots, n(\tau, N_m) + qn'_m$, would render quasi-arbitrage, as $m \to \infty$. The reader is invited to prove the first inequality in (20) by making use of the inequality $q\hat{U}_{n,N_m}(1) \geq \hat{U}_{n,N_m}(q) + \frac{\varepsilon}{3}$ if $n(\tau, N_m) \leq n \leq N_m - n(\tau, N_m)$. The proposition follows now from the maintained assumptions. $\square$

Proof of multi-asset versions of Propositions 1 and 2. The proof is divided into five steps. $\hat{U}_{ij}$, $i \neq j$, is as defined in the main text. We first prove the necessity of U's quasi-linearity in the absence of price manipulation and δ-arbitrage. Quasi-arbitrage is discussed at the end.

Step 1: $\hat{U}_{ij}$ is symmetric, i.e., $\hat{U}_{ij}(q) = -\hat{U}_{ij}(-q)$ on $\mathcal{D}^j_M$.

If not, then either (i) there exists $q > 0$ with $\hat{U}_{ij}(q) > -\hat{U}_{ij}(-q)$, or (ii) there exists $q > 0$ with $\hat{U}_{ij}(q) < -\hat{U}_{ij}(-q)$, or (iii) $\hat{U}_{ij}(0) \neq 0$. For case (i) consider the strategy of buying q shares of asset i in each of the first m periods, buying q shares of asset j in each of the next m periods, selling q shares of asset j in each of the next m periods, and selling q shares of asset i in each of the following m periods. This implies $\mathbb{E}[\pi^0_{4m}] = O(m^2 q[\hat{U}_{ij}(q) + \hat{U}_{ij}(-q)])$. For case (ii) consider selling in each of the first m periods q shares of asset i, buying in each of the next m periods q shares of asset j, selling in each of the next m periods q shares of asset j, and buying in each of the subsequent m periods q shares of asset i. Then, $\mathbb{E}[\pi^0_{4m}] = O(-m^2 q[\hat{U}_{ij}(q) + \hat{U}_{ij}(-q)])$. Both trading strategies give $\mathbb{E}[\pi^0_{4m}] > 0$ and a Sharpe ratio that can become larger than any δ bound if the initial price vector is sufficiently high. Case (iii) is easy to rebut and left to the reader.

Step 2: $\lim_{n\to\infty} \hat{U}_{ij}(q^{(q)}_n) = \hat{U}_{ij}(q)$ when $q \in \mathcal{D}^j_M \backslash \{0\}$ is irrational and $\lim_{n\to\infty} q^{(q)}_n = q$, all $q^{(q)}_n \in \mathbf{Q}$. In what follows, we verify that none of the four cases stated in Lemma 2 can hold for $\hat{U}_{ij}$. For the first case take the strategy of buying q shares of asset i in each of the first m periods, buying $q^{(q)}_n$ shares of asset j in each of the next m periods, selling q shares of asset j in each of the next m periods, and selling q shares of asset i in each of the following m periods. This results in $\mathbb{E}[\pi^0_{4m}] = O(m^2 q[\hat{U}_{ij}(q^{(q)}_n) - \hat{U}_{ij}(q)])$. For the second case, consider buying q shares of asset i in each of the first m periods, buying q shares of asset j in each of the next m periods, selling $q^{(q)}_n$ shares of asset j in each of the following $m-1$ periods, and selling q shares of asset i in each of the next m periods. As a consequence, $\mathbb{E}[\pi^0_{4m-1}] = O(-m^2[\hat{U}_{ji}(q)(q - q^{(q)}_n) + \frac{1}{2}q^{(q)}_n \hat{U}_{jj}(q^{(q)}_n) + \frac{1}{2}q\hat{U}_{jj}(q)(q - 2q^{(q)}_n) + q[\hat{U}_{ij}(q^{(q)}_n) - \hat{U}_{ij}(q)]])$. In the third case, take the strategy of buying q shares of asset i in each of the first m periods, buying $q^{(q)}_n$ shares of asset j in each of the next m periods, selling q shares of asset j in each of the next $m-1$ periods, and selling q shares of asset i in each of the following m periods. We obtain $\mathbb{E}[\pi^0_{4m-1}] = O(-m^2[\hat{U}_{ji}(q)(q^{(q)}_n - q) + \frac{1}{2}\hat{U}_{jj}(q^{(q)}_n)(q^{(q)}_n - 2q) + \frac{1}{2}q\hat{U}_{jj}(q) + q[\hat{U}_{ij}(q) - \hat{U}_{ij}(q^{(q)}_n)]])$. For the last case, consider buying q shares of asset i in each of the first m periods, buying q shares of asset j in each of the next m periods, selling $q^{(q)}_n$ shares of asset j in each of the following m periods, and selling q shares of asset i in each of the next m periods. This yields $\mathbb{E}[\pi^0_{4m}] = O(-m^2[\hat{U}_{ji}(q)(q - q_n^{(q)}) + \hat{U}_{jj}(q)(\frac{1}{2}q - q_n^{(q)}) + \frac{1}{2}q_n^{(q)}\hat{U}_{jj}(q_n^{(q)}) - q[\hat{U}_{ij}(q) - \hat{U}_{ij}(q_n^{(q)})]])$. All trading strategies render positive expected profits and high Sharpe ratios when m and the index n are chosen appropriately.

Step 3: $\hat{U}_{ij}(q) = \hat{U}_{ij}(1)q$ on $\mathcal{D}^j_M$. If it were not, either $\hat{U}_{ij}(q) > \hat{U}_{ij}(1)q$ for a $q > 0$ or $\hat{U}_{ij}(q) < \hat{U}_{ij}(1)q$ for a $q > 0$. In the first case, the trading strategy of buying q shares of asset i in the first m periods, buying q shares of asset j in the next m periods, selling one share of asset j in each of the next mq periods, and selling q shares of asset i in each of the next m periods gives $\mathbb{E}[\pi^0_{4m}] = O(m^2 q[\hat{U}_{ij}(q) - \hat{U}_{ij}(1)q])$. In the second case, we obtain $\mathbb{E}[\pi^0_{4m}] = O(-m^2 q[\hat{U}_{ij}(q) - \hat{U}_{ij}(1)q])$ from buying q shares of asset i in each of the first m periods, buying one share of asset j in each of the next mq periods, selling q shares of asset j in each of the next m periods, and buying q shares of asset i in each of the following m periods. Both are at variance with the absence of price manipulation and δ-arbitrage if m is large enough.

Step 4: $\hat{U}_{ij} = \hat{U}_{ji}$. Consider the strategy of buying q shares of asset i in each of the first m periods, buying q shares of asset j in each of the next m periods, selling q shares of asset i in each of the next m periods, and selling q shares of asset j in each of the next m periods. This implies $\mathbb{E}[\pi^0_{4m}] = O(m^2 q[\hat{U}_{ij}(q) - \hat{U}_{ji}(q)])$. Obviously, this is in discord with the absence of price manipulation and δ-arbitrage if $\hat{U}_{ij}(q) > \hat{U}_{ji}(q)$ for a $q > 0$. By the same arguments, the opposite inequality, i.e., $\hat{U}_{ij}(q) < \hat{U}_{ji}(q)$ for a $q > 0$ is false, too.

Last Step: $\hat{U}_i(q_1, q_2, \ldots, q_K) = \sum_{j=1}^K \hat{U}_{ij}(q_j)$. For brevity we prove the latter equality only for the case $K = 2$ here; the extension to arbitrary K is straightforward. Take m even and employ the following two strategies.

1. Strategy $\mathtt{X}$: trade $-q$ shares of asset j in each of the first $\frac{m}{2}$ periods, trade q shares each of asset i and asset j in each of the next m periods, trade q shares of asset j in each of the next $\frac{m}{2}$ periods, trade $-q$ shares of asset j in each of the following m periods, and trade $-q$ shares of asset i in each of the next m periods;

2. Strategy $\mathtt{Y}$: trade $-q$ shares of asset j in each of the first m periods, trade $-q$ shares of asset i in each of the next m periods, trade q shares of asset j in each of the following $\frac{m}{2}$ periods, trade q shares each of asset i and asset j in each of the next m periods, and trade $-q$ shares of asset j in each of the next $\frac{m}{2}$ periods.

Strategy $\mathtt{X}$ gives rise to $\mathbb{E}[\pi^0_{4m}] = O(m^2 q_i[\hat{U}_i(q_1,q_2) - \hat{U}_{ii}(q_i) - \hat{U}_{ij}(q_j)])$, while strategy $\mathtt{Y}$ has $\mathbb{E}[\pi^0_{4m}] = O(-m^2 q_i[\hat{U}_i(q_1,q_2) - \hat{U}_{ii}(q_i) - \hat{U}_{ij}(q_j)])$ as a result. Hence, regardless of the value of q_i, the absence of price manipulation and δ-arbitrage implies $\hat{U}_i(q_1,q_2) = \hat{U}_{ii}(q_i) + \hat{U}_{ij}(q_j)$. Hence, $\hat{U}_i$ is linear.

The slope λ of U has to be positive semidefinite: if there exists a q such that $q^T\lambda q < 0$, then consider the strategy of trading the vector q in each of the first m periods, and then trading the vector $-q$ in each of the following m periods. The result is $\mathbb{E}[\pi^0_{2m}] = O(-m^2 q^T \lambda q)$ and hence λ has to be positive semidefinite to rule price manipulation and δ-arbitrage. That U is quasi-linear can be shown in the same fashion as in the single-asset case (see proof of Proposition 1).

To show that the absence of quasi-arbitrage implies the quasi-linearity of U, we employ the fact that each trading strategy used in Steps 2-4 causes the expected final prices (after the implementation of each trading strategy) to be at least as large as the initial prices. Thus, if each trading strategy in Steps 2-4 is repeated infinitely many times, expected prices always stay nonnegative, and the expected value and the Sharpe ratio of each strategy's profits go to infinity. As a consequence, the absence of quasi-arbitrage puts the same restrictions on the function $\hat{U}_{ij}$ as the absence of price manipulation does. To verify the linearity of $\hat{U}_i$, use of the symmetry assumption $\hat{U}_i(q_1,q_2) = -\hat{U}_i(-q_1,-q_2)$ is made. We have to distinguish twelve cases:

1. if $\hat{U}_i(q_1,q_2) > \hat{U}_{ii}(q_i) + \hat{U}_{ij}(q_j)$, $\hat{U}_j(q_1,q_2) = \hat{U}_{jj}(q_j) + \hat{U}_{ji}(q_i)$, and $q_i > 0$, repeat strategy $\mathtt{X}$ infinitely many times;

2. if $\hat{U}_i(q_1,q_2) > \hat{U}_{ii}(q_i) + \hat{U}_{ij}(q_j)$, $\hat{U}_j(q_1,q_2) = \hat{U}_{jj}(q_j) + \hat{U}_{ji}(q_i)$, and $q_i < 0$, repeat strategy $\mathtt{Y}$ infinitely many times;

3. if $\hat{U}_i(q_1, q_2) < \hat{U}_{ii}(q_i) + \hat{U}_{ij}(q_j)$, $\hat{U}_j(q_1, q_2) = \hat{U}_{jj}(q_j) + \hat{U}_{ji}(q_i)$, and $q_i > 0$, use strategy `Y` for the case $\hat{U}_i(-q_1, -q_2) > \hat{U}_{ii}(-q_i) + \hat{U}_{ij}(-q_j)$ and $-q_i < 0$ infinitely many times;

4. if $\hat{U}_i(q_1, q_2) < \hat{U}_{ii}(q_i) + \hat{U}_{ij}(q_j)$, $\hat{U}_j(q_1, q_2) = \hat{U}_{jj}(q_j) + \hat{U}_{ji}(q_i)$, and $q_i < 0$, use strategy `X` for the case $\hat{U}_i(-q_1, -q_2) > \hat{U}_{ii}(-q_i) + \hat{U}_{ij}(-q_j)$ and $-q_i > 0$ infinitely many times;

5. if $\hat{U}_i(q_1, q_2) > \hat{U}_{ii}(q_i) + \hat{U}_{ij}(q_j)$, $\hat{U}_j(q_1, q_2) > \hat{U}_{jj}(q_j) + \hat{U}_{ji}(q_i)$, and $q_i > 0$, repeat strategy `X` infinitely many times;

6. if $\hat{U}_i(q_1, q_2) > \hat{U}_{ii}(q_i) + \hat{U}_{ij}(q_j)$, $\hat{U}_j(q_1, q_2) > \hat{U}_{jj}(q_j) + \hat{U}_{ji}(q_i)$, and $q_i < 0$, repeat strategy `Y` infinitely many times;

7. if $\hat{U}_i(q_1, q_2) < \hat{U}_{ii}(q_i) + \hat{U}_{ij}(q_j)$, $\hat{U}_j(q_1, q_2) > \hat{U}_{jj}(q_j) + \hat{U}_{ji}(q_i)$, and $q_i > 0$, first use strategy `Y` and then use strategy `Y` for the case $\hat{U}_i(-q_1, -q_2) > \hat{U}_{ii}(-q_i) + \hat{U}_{ij}(-q_j)$, $\hat{U}_j(-q_1, -q_2) < \hat{U}_{jj}(-q_j) + \hat{U}_{ji}(-q_i)$, and $-q_i < 0$, which we denote by `-Y`; repeat this pair of trading infinitely many times, i.e., trade `Y`, `-Y`,`Y`, `-Y`, `Y`, and so on;

8. if $\hat{U}_i(q_1, q_2) < \hat{U}_{ii}(q_i) + \hat{U}_{ij}(q_j)$, $\hat{U}_j(q_1, q_2) > \hat{U}_{jj}(q_j) + \hat{U}_{ji}(q_i)$, and $q_i < 0$, first use strategy `X` and then use strategy `X` for the case $\hat{U}_i(-q_1, -q_2) > \hat{U}_{ii}(-q_i) + \hat{U}_{ij}(-q_j)$, $\hat{U}_j(-q_1, -q_2) < \hat{U}_{jj}(-q_j) + \hat{U}_{ji}(-q_i)$, and $-q_i > 0$, which we denote by `-X`; trade infinitely many times `X`, `-X`,`X`, `-X`, `X`, and so on;

9. if $\hat{U}_i(q_1, q_2) > \hat{U}_{ii}(q_i) + \hat{U}_{ij}(q_j)$, $\hat{U}_j(q_1, q_2) < \hat{U}_{jj}(q_j) + \hat{U}_{ji}(q_i)$, and $q_i > 0$, first use strategy `X` and then use either strategy `X` or `Y` for the case $\hat{U}_j(-q_1, -q_2) > \hat{U}_{jj}(-q_j) + \hat{U}_{ji}(-q_i)$, $\hat{U}_i(-q_1, -q_2) < \hat{U}_{ii}(-q_i) + \hat{U}_{ij}(-q_j)$ (asset i is traded first), depending on whether $-q_j > 0$ or $-q_j < 0$, respectively; repeat this pair of trading infinitely many times;

10. if $\hat{U}_i(q_1, q_2) > \hat{U}_{ii}(q_i) + \hat{U}_{ij}(q_j)$, $\hat{U}_j(q_1, q_2) < \hat{U}_{jj}(q_j) + \hat{U}_{ji}(q_i)$, and $q_i < 0$, first use strategy `Y` and then use either strategy `X` or `Y` for the case $\hat{U}_j(-q_1, -q_2) > \hat{U}_{jj}(-q_j) + \hat{U}_{ji}(-q_i)$, $\hat{U}_i(-q_1, -q_2) < \hat{U}_{ii}(-q_i) + \hat{U}_{ij}(-q_j)$ (asset i is traded first), depending on whether $-q_j > 0$ or $-q_j < 0$, respectively; repeat this pair of trading infinitely many times;

11. if $\hat{U}_i(q_1, q_2) < \hat{U}_{ii}(q_i) + \hat{U}_{ij}(q_j)$, $\hat{U}_j(q_1, q_2) < \hat{U}_{jj}(q_j) + \hat{U}_{ji}(q_i)$, and $q_i > 0$, use infinitely many times strategy Y for the case $\hat{U}_i(-q_1, -q_2) > \hat{U}_{ii}(-q_i) + \hat{U}_{ij}(-q_j)$, $\hat{U}_j(-q_1, -q_2) > \hat{U}_{jj}(-q_j) + \hat{U}_{ji}(-q_i)$, and $-q_i < 0$;

12. if $\hat{U}_i(q_1, q_2) < \hat{U}_{ii}(q_i) + \hat{U}_{ij}(q_j)$, $\hat{U}_j(q_1, q_2) < \hat{U}_{jj}(q_j) + \hat{U}_{ji}(q_i)$, and $q_i < 0$, use infinitely many times strategy X for the case $\hat{U}_i(-q_1, -q_2) > \hat{U}_{ii}(-q_i) + \hat{U}_{ij}(-q_j)$, $\hat{U}_j(-q_1, -q_2) > \hat{U}_{jj}(-q_j) + \hat{U}_{ji}(-q_i)$, and $-q_i > 0$.

Each trading strategy described for the cases 1-12 produces infinite expected profits, while expected prices always stay nonnegative. In addition, the Sharpe ratios go to infinity thanks to the assumption made in the main text ($\gamma < 0$, $\zeta < 0$, and $\vartheta < 0$). Therefore, $\hat{U}_i$ has to be linear. Moreover, if there exists a q such that $q^T \lambda q < 0$, then consider the strategy of trading the vector q in each of the first m periods, and then trading the vector $-q$ in each of the following m periods. Then, reverse the roles and trade the vector $-q$ in each of the subsequent m periods, and then trade the vector q in each of the following m periods. If you repeat this $4m$ periods long round-trip trade infinitely many times you obtain $\lim_{m\to\infty} \mathbb{E}[\pi^0_{2m}] = \lim_{m\to\infty} SR[\pi^0_{2m}] = \infty$ with expected prices always being nonnegative. As a consequence, λ must be positive semidefinite. To complete, use the same arguments as in the proof of Proposition 1 to show that the absence of quasi-arbitrage implies the quasi-linearity of U. $\square$

Appendix B. Examples of Nonzero Supplementary Functions

We give here three examples of quasi-linear functions whose supplementary functions fail to be zero. The proofs are presented after a brief discussion of these examples.

Example A (Bernoulli distribution) Suppose $\mathcal{D}_M = \mathbf{R}$ and that the residual trades can only assume two values with positive probability, namely, $\mathbb{P}[\eta_n = -\eta_0] = \mathbb{P}[\eta_n = \eta_0] = \frac{1}{2} = \frac{1}{2} + \mathbb{P}[\eta_n = 0]$ for $n \in N$ and $\eta_0 > 0$. In this case, U is quasi-linear if and only if $U(x) = \lambda x + S_U(x)$ where $S_U(x) = -S_U(x - 2\eta_0)$ for all $x \in \mathbf{R}$.

Example B (Uniform distribution) Assume $\mathcal{D}_M = \mathbf{R}$ and that the η_n's are uniformly distributed on $\mathbf{R}$, with compact support $[-s, s]$, $s > 0$, and that U is either continuous and of bounded variation or piecewise continuously differentiable on $\mathbf{R}$. Then, U is quasi-linear if and only if $U(x) = \lambda x + S_U(x)$ where S_U is a $2s$-periodic trigonometric Fourier series satisfying $\int_0^{2s} S_U(x)dx = 0$. (For the precise form of S_U see below.)

Example C (Triangle distribution) Let the residual trades have the "triangle density"

$$f_\eta(x) = \begin{cases} (1 + \frac{x}{s})/s & x \in [-s, 0] \\ (1 - \frac{x}{s})/s & x \in (0, s] \end{cases}, \quad s > 0,$$

on $\mathbf{R}$. In this case, U is quasi-linear if and only if $U(x) = \lambda x + S_U(x)$ where S_U is given by

$$S_U(q) = S_1(q) + S_2(q)q, \tag{21}$$

where $S_1 : \mathbf{R} \to \mathbf{R}$ and $S_2 : \mathbf{R} \to \mathbf{R}$ are s-periodic functions satisfying $\int_0^s S_1(q)dq = \int_0^s S_2(q)dq = 0$.

Observe that to derive the result in Example B, we need to impose smoothness assumptions on U, unlike the results in Propositions 2 and 5 and Examples A and C.

In Examples A, B, and C, S_U can take on a variety of functional forms. What they have in common is that they are periodic and that either S_U or its components integrate to zero over any interval with length equal to their periodicity. For instance, any multiple of the sine function would be a possible candidate for the functions S_U, S_1, and S_2 in Examples B and C, if the periodicity is $s = \pi$ and $s = 2\pi$, respectively. The reader is invited to construe candidate S_U-functions for Example A.

Note that the precise shape of S_U is determined by the curvature of the residual trades' density function and is therefore variable. For more complicated distributions, S_U can still be computed, albeit with much more intricate structure.

For the proofs recall from above that $\int_\Omega S_U(q+\eta_n)d\mathbb{P} = 0$ for all $q \in \mathcal{D}_M$ is equivalent to $\mathbb{E}_n[S_U(\tilde{q}_n +$

$\eta_n)] = 0$, for any $\mathcal{F}_n$-measurable random variable $\tilde{q}_n$.

Proof of Example A. By definition, quasi-linearity requires the supplementary function of U to meet

$$S_U(q+\eta_0) + S_U(q-\eta_0) = 0 \quad \text{for all } q \in \mathbf{R}. \tag{22}$$

On the other hand, (22) implies that $U(x) = \lambda x + S_U(x)$ is quasi-linear. □

Proof of Example B. Under the assumption that U is quasi-linear, equation (6) becomes

$$\int_{q-s}^{q+s} S_U(x)dx = 0 \quad \text{for all } q \in \mathbf{R}. \tag{23}$$

By differentiating the above integral equation with respect to q, we obtain that S_U is $2s$-periodic on $\mathbf{R}$. Since, by assumption, S_U is either continuous and of bounded variation or piecewise continuously differentiable, it has a trigonometric Fourier representation given by

$$S_U(q) = \sum_{n=1}^{\infty} \left[a_n \cos(\frac{\pi s}{n} q) + b_n \sin(\frac{\pi s}{n} q) \right], \tag{24}$$

where

$$a_n = \frac{1}{s} \int_{-s}^{s} S_U(x) \cos(\frac{\pi s}{n} x)dx \tag{25}$$

and

$$b_n = \frac{1}{s} \int_{-s}^{s} S_U(x) \sin(\frac{\pi s}{n} x)dx. \tag{26}$$

Note that the above Fourier series does not have an intercept part, a_0, since $a_0 = \int_{-s}^{s} S_U(x)dx = 0$. Hence S_U possesses the claimed $2s$-periodic trigonometric Fourier series with $\int_0^{2s} S_U(x)dx = 0$.

Conversely, if $S_U(q) = U(q) - \lambda q$ is $2s$-periodic and satisfies $\int_0^{2s} S_U(x)dx = 0$, then it has the representation (24)-(26) and meets (23). Thus U is quasi-linear and the proof is complete. □

Proof of Example C. We first show that in case of quasi-linearity S_U is given by (21), where S_1 :

$\mathbf{R} \to \mathbf{R}$ and $S_2 : \mathbf{R} \to \mathbf{R}$ are s-periodic functions.

If the density of the residual trades is f_η, equation (6) has the form

$$\int_{q-s}^{q+s} S_U(x) f_\eta(x-q)dx = 0 \quad \text{for all } q \in \mathbf{R}. \tag{27}$$

Now, differentiating this integral equation with respect to q yields

$$\int_{q-s}^{q} S_U(x)dx - \int_{q}^{q+s} S_U(x)dx = 0 \quad \text{for all } q \in \mathbf{R}.$$

By differentiating again, we obtain that S_U satisfies the difference equation

$$S_U(q+2s) - 2S_U(q+s) + S_U(q) = 0 \qquad \text{for all } q \in \mathbf{R}.$$

But the general solution of it is just given by (21), where S_1 and S_2 are both s-periodic.

That S_1 and S_2 satisfy the two integral conditions stated in Example C follows from the identity

$$\int_{q-s}^{q+s} S_U(x) f_\eta(x-q)dx =$$

$$\frac{1}{s}\int_{q}^{q+s} S_1(x)dx + \frac{1}{s}\int_{q}^{q+s} S_2(x)dx * q = 0$$

for all $q \in \mathbf{R}$.

Alternately, the last equation implies that $S_U(q) = U(q) - \lambda q$ satisfies (27) if it has the representation (21) with $\int_0^s S_1(q)dq = \int_0^s S_2(q)dq = 0$. Therefore, U is quasi-linear. □

Appendix C. Proofs of the results in Section VII

Proof of Propositions 1, 2*, 5*, 9*, and 10*.* Propositions 1* and 2* are shown in exactly the same fashion as Propositions 1 and 2, that means, using the same steps and the same trading strategies.

To evaluate the gain-loss ratio use in each case the inequality $\mathbb{E}[(X+Y)^+]/\mathbb{E}[(X+Y)^-] \geq (\mathbb{E}[X] + \mathbb{E}[Y])/(\mathbb{E}[X^-] + \mathbb{E}[Y^-])$ for any two random variables $X : \Omega \rightarrow \mathbf{R}$ and $Y : \Omega \rightarrow \mathbf{R}$. Proposition 5* is easier to prove than its counterpart Proposition 5. As straightforward computations reveal, the revenue of a multi-period round-trip trade can be written as $\pi_N^0 = X + Y$, where $X \leq 0$ and $\mathbb{E}[Y] = 0$. By virtue of

$$\frac{\mathbb{E}[(\pi_N^0)^+]}{\mathbb{E}[(\pi_N^0)^-]} \leq \frac{\mathbb{E}[Y^+]}{\mathbb{E}[Y^-]} = 1,$$

we conclude that quasi-linearity of the price-update and price-impact functions implies $\mathrm{GLR}[\pi_N^0] \leq 1$. Finally, Propositions 9* and 10* can be verified by repeating the proofs of Propositions 9 and 10, where instead of the Sharpe ratio the gain-loss ratio has to be calculated at each step. $\Box$

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