Stochastic Portfolio Theory: an Overview

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Summary

Stochastic Portfolio Theory is a flexible framework for analyzing portfolio behavior and equity market structure. This theory was introduced by E.R. Fernholz in the papers (*Journal of Mathematical Economics*, 1999; *Finance & Stochastics*, 2001) and in the monograph *Stochastic Portfolio Theory* (Springer 2002). It was further developed in the papers Fernholz, Karatzas & Kardaras (*Finance & Stochastics*, 2005), Fernholz & Karatzas (*Annals of Finance*, 2005), Banner, Fernholz & Karatzas (*Annals of Applied Probability*, 2005), and Karatzas & Kardaras (2006). This theory is descriptive, as opposed to normative; it is consistent with observable characteristics of actual portfolios and markets; and it provides a theoretical tool which is useful for practical applications.

As a theoretical tool, this framework offers fresh insights into questions of stock market structure and arbitrage, and can be used to construct portfolios with controlled behavior. As a practical tool, Stochastic Portfolio Theory has been applied to the analysis and optimization of portfolio performance and has been the basis of successful investment strategies for over a decade.

Subject Index

Introduction

Stochastic Portfolio Theory (SPT), as we currently think of it, began in 1995 with the manuscript "On the diversity of equity markets", which eventually appeared as Fernholz (1999) in the *Journal of Mathematical Economics*. Since then SPT has evolved into a flexible framework for analyzing portfolio behavior and equity market structure that has both theoretical and practical applications. As a theoretical methodology, this framework provides insight into questions of market behavior and arbitrage, and can be used to construct portfolios with controlled behavior under quite general conditions. As a practical tool, SPT has been applied to the analysis and optimization of portfolio performance and has been the basis of successful equity investment strategies for over a decade.

SPT is a *descriptive theory*, which studies and attempts to explain observable phenomena that take place in equity markets. This orientation is quite different from that of the well-known *Modern Portfolio Theory of Dynamic Asset Pricing (DAP),* in which market structure is analyzed under strong normative assumptions regarding the behavior of market participants. It has long been suggested that the distinction between descriptive and normative theories separates the natural sciences from the social sciences; if this dichotomy is valid, then one might argue that SPT resides with the natural sciences.

SPT descends from the "classical portfolio theory" of Harry Markowitz (1952), as does much of mathematical finance. At the same time, it represents a rather significant departure from some important aspects of the current theory of Dynamic Asset Pricing. DAP is a normative theory that grew out of the general equilibrium model of mathematical economics for financial markets, evolved through the capital asset pricing models, and is currently predicated on the absence of arbitrage and on the existence of equivalent martingale measure(s). SPT, by contrast, is applicable under a wide range of assumptions and conditions that may hold in actual equity markets. Unlike dynamic asset pricing, it is consistent with either equilibrium or disequilibrium, with either arbitrage or no-arbitrage, and is not predicated on the existence of equivalent martingale measure(s).

While SPT has been developed with equity markets in mind, a reasonable portion of the theory is valid for general financial assets, as long as the asset values remain positive. For such general assets, the "market" can be replaced by an arbitrary passive portfolio with positive holdings in each of the assets. Although some concepts related to equity markets may not be meaningful in these general applications, others would appear to carry over without significant modification.

This survey reviews the central ideas of SPT and presents examples of portfolios and markets with a wide variety of different properties. SPT is a fast-evolving field, so we also present a number of research problems that remain open, at least at the time of this writing. Proofs for some of the results are included here, but at other times simply a reference is given.

The survey is separated into four chapters. Chapter I, Basics, introduces the concepts of markets and portfolios, in particular, the *market portfolio*, that most important portfolio of them all. In this first chapter we also encounter the *excess growth rate* process, a quantity that pervades SPT. Chapter II, Diversity & Arbitrage, introduces market *diversity* and shows how diversity can lead to *relative arbitrage* in an equity market. Historically, these were among the first phenomena analyzed using SPT. *Portfolio generating functions* are versatile tools for constructing portfolios with particular properties, and these functions are discussed in Chapter III, Functionally Generated Portfolios. Here we also consider stocks identified by rank, as opposed to by name, and discuss implications regarding the *size effect*. Roughly speaking, these first three chapters of the survey outline the techniques that historically have comprised SPT; the fourth chapter looks toward the future.

Part IV, Abstract Markets, is devoted to the area of much of the current research in SPT. *Abstract markets* are models of equity markets that exhibit certain characteristics of real stock markets, but for which the precise mathematical structure is known (since we can define them as we wish!). Here we see *volatility-stabilized* markets that are not diverse but nevertheless allow arbitrage, and we also look at *rank-based* markets that have stability properties similar to those of real stock markets. Several problems regarding these abstract markets are proposed.

Chapter I
Basics

SPT uses the *logarithmic representation* for stocks and portfolios rather than the *arithmetic representation* used in "classical" mathematical finance. In the logarithmic representation, the classical *rate of return* is replaced by the *growth rate*, sometimes referred to as the *geometric rate of return* or the *logarithmic rate of return.* The logarithmic and arithmetic representations are equivalent, but nevertheless the different perspectives bring to light distinct aspects of portfolio behavior. The use of the logarithmic representation in no way implies the use of a logarithmic utility function: indeed, SPT is not concerned with expected utility maximization at all.

We introduce here the basic structures of SPT, *stocks* and *portfolios*, and discuss that most important portfolio of them all: the *market portfolio.* We show that the growth rate of a portfolio depends not only on the growth rates of the component stocks, but also on the *excess growth rate*, which is determined by the stocks' variances and covariances. Finally, we consider a few optimization problems in the logarithmic setting.

Most of the material in this chapter can be found in Fernholz (2002).

1 Markets and Portfolios

We shall place ourselves in a model $\mathcal{M}$ for a financial market of the form

$$\begin{aligned} dB(t) &= B(t)r(t)\,dt, \qquad\qquad B(0)=1, \\ dX_i(t) &= X_i(t)\Big(b_i(t)dt + \sum_{\nu=1}^{d}\sigma_{i\nu}(t)\,dW_\nu(t)\Big), \qquad X_i(0)=x_i>0,\ i=1,\dots,n, \end{aligned} \tag{1.1}$$

consisting of a money-market $B(\cdot)$ and of n stocks, whose prices $X_1(\cdot),\cdots,X_n(\cdot)$ are driven by the $d-$dimensional Brownian motion $W(\cdot)=\big(W_1(\cdot),\cdots,W_d(\cdot)\big)'$ with $d\ge n$. Contrary to a usual assumption imposed on such models, here it is not crucial that the filtration $\mathbb{F}=\{\mathcal{F}(t)\}_{0\le t<\infty}$, which represents the "flow of information" in the market, be the one generated by the Brownian motion itself. Thus, and until further notice, we shall take $\mathbb{F}$ to contain (possibly strictly) this Brownian filtration $\mathbb{F}^W=\{\mathcal{F}^W(t)\}_{0\le t<\infty}$, where $\mathcal{F}^W(t):=\sigma(W(s),\,0\le s\le t)\subseteq\mathcal{F}(t)\,,\ \ \forall\ t\in[0,\infty)\,.$

We shall assume that the interest-rate process $r(\cdot)$ for the money-market, the vector-valued process $b(\cdot)=\big(b_1(\cdot),\dots,b_n(\cdot)\big)'$ of *rates of return* for the various stocks, and the $(n\times d)-$matrix-valued process $\sigma(\cdot)=\big(\sigma_{i\nu}(\cdot)\big)_{1\le i\le n,\,1\le\nu\le d}$ of stock-price *volatilities*, are all $\mathbb{F}-$progressively measurable and satisfy for every $T\in(0,\infty)$ the integrability conditions

$$\int_0^T |r(t)|\,dt + \sum_{i=1}^n\int_0^T\Big(\big|b_i(t)\big| + \sum_{\nu=1}^d\big(\sigma_{i\nu}(t)\big)^2\Big)\,dt<\infty\,, \qquad \text{a.s.} \tag{1.2}$$

This setting admits a rich class of continuous-path Itô processes, with very general distributions: in particular, *no Markovian or Gaussian assumption is imposed.* In fact, it is possible to extend the scope of the theory to very general semimartingale settings; see Kardaras (2003) for details.

We shall introduce the notation

$$a_{ij}(t):=\sum_{\nu=1}^d\sigma_{i\nu}(t)\sigma_{j\nu}(t)=\big(\sigma(t)\sigma'(t)\big)_{ij}=\frac{d}{dt}\langle\log X_i,\log X_j\rangle(t) \tag{1.3}$$

for the non-negative definite matrix-valued *covariance process* $a(\cdot)=\big(a_{ij}(\cdot)\big)_{1\le i,j\le n}$ of the stocks in the market, as well as

$$\gamma_i(t):=b_i(t)-\frac{1}{2}a_{ii}(t), \qquad i=1,\dots,n\,. \tag{1.4}$$

Then we may use Itô's rule to solve (1.1) in the form

$$d \log X_i(t) \;=\; \gamma_i(t)\, dt + \sum_{\nu=1}^{d} \sigma_{i\nu}(t)\, dW_\nu(t), \quad i = 1, \ldots, n, \tag{1.5}$$

or equivalently:

$$X_i(t) = x_i \, \exp\Big\{ \int_0^t \gamma_i(u)\, du + \sum_{\nu=1}^{d} \int_0^t \sigma_{i\nu}(u)\, dW_\nu(u) \Big\}, \quad 0 \le t < \infty.$$

Equation (1.5) is called the *logarithmic representation* of the stock price process, and we shall refer to the quantity of (1.4) as the *growth rate* of the i^{th} stock, because of the a.s. relationship

$$\lim_{T\to\infty} \frac{1}{T} \Big(\log X_i(T) - \int_0^T \gamma_i(t)\, dt \Big) = 0\,. \tag{1.6}$$

This is valid when the individual stock variances $a_{ii}(\cdot)$ do not increase too quickly, e.g., if we have

$$\lim_{T\to\infty} \left(\frac{\log \log T}{T^2} \int_0^T a_{ii}(t)\, dt \right) \;=\; 0\,, \qquad \text{a.s.}; \tag{1.7}$$

then (1.6) follows from the the iterated logarithm and from the representation of (local) martingales as time-changed Brownian motions.

Definition 1.1. A *portfolio* $\pi(\cdot) \;=\; \big(\pi_1(\cdot), \ldots,\; \pi_n(\cdot)\big)'$ is an $\mathbb{F}-$progressively measurable process, bounded uniformly in (t, ω), with values in the set

$$\bigcup_{\kappa\in\mathbb{N}} \big\{ (\pi_1, \ldots, \pi_n) \in \mathbb{R}^n \;\big|\; \pi_1^2 + \ldots + \pi_n^2 \le \kappa^2\,,\; \pi_1 + \cdots + \pi_n = 1 \big\}.$$

A *long-only portfolio* $\pi(\cdot) = \big(\pi_1(\cdot), \ldots, \pi_n(\cdot)\big)'$ is a portfolio that takes values in the set

$$\Delta^n := \big\{ (\pi_1, \ldots, \pi_n) \in \mathbb{R}^n \;\big|\; \pi_1 \ge 0, \ldots, \pi_n \ge 0 \quad \text{and} \quad \pi_1 + \cdots + \pi_n = 1 \big\}.$$

For future reference, we shall introduce also the notation

$$\Delta^n_+ := \big\{ (\pi_1, \ldots, \pi_n) \in \Delta^n \,|\, \pi_1 > 0, \ldots, \pi_n > 0 \big\}. \qquad \Box$$

Thus, a portfolio can sell one or more stocks short (though certainly not all) but is never allowed to borrow from, or invest in, the money market; whereas a long-only portfolio sells no stocks short at all. The interpretation is that $\pi_i(t)$ represents the proportion of wealth $V^{w,\pi}(t)$ invested at time t in the i^{th} stock, so the quantities

$$h_i(t) = \pi_i(t) V^{w,\pi}(t), \qquad i = 1, \cdots, n \tag{1.8}$$

are the dollar amounts invested at any given time t in the individual stocks.

The wealth process $V^{w,\pi}(\cdot)$, that corresponds to a portfolio $\pi(\cdot)$ and initial capital $w > 0$, satisfies the stochastic equation

$$\begin{aligned} \frac{dV^{w,\pi}(t)}{V^{w,\pi}(t)} \;&=\; \sum_{i=1}^{n} \pi_i(t)\, \frac{dX_i(t)}{X_i(t)} \;=\; \pi'(t) \big[\, b(t)dt + \sigma(t)\, dW(t) \,\big] \\ &=\; b_\pi(t)\, dt + \sum_{\nu=1}^{d} \sigma_{\pi\nu}(t)\, dW_\nu(t), \quad V^{w,\pi}(0) = w, \end{aligned} \tag{1.9}$$

where

$$b_\pi(t) := \sum_{i=1}^n \pi_i(t) b_i(t), \qquad \sigma_{\pi\nu}(t) := \sum_{i=1}^n \pi_i(t)\sigma_{i\nu}(t) \quad \text{for} \quad \nu = 1, \ldots, d\,. \tag{1.10}$$

These quantities are, respectively, the rate-of-return and the volatility coëfficients associated with the portfolio $\pi(\cdot)$.

By analogy with (1.5) we can write the solution of the equation (1.9) as

$$d \log V^{w,\pi}(t) = \gamma_\pi(t)\, dt + \sum_{\nu=1}^d \sigma_{\pi\nu}(t)\, dW_\nu(t), \qquad V^{w,\pi}(0) = w\,, \tag{1.11}$$

or equivalently

$$V^{w,\pi}(t) = w\, \exp\Big\{ \int_0^t \gamma_\pi(u)\, du + \sum_{\nu=1}^d \int_0^t \sigma_{\pi\nu}(u)\, dW_\nu(u) \Big\}\,, \quad 0 \le t < \infty\,.$$

Here

$$\gamma_\pi(t) := \sum_{i=1}^n \pi_i(t)\gamma_i(t) + \gamma_\pi^*(t) \tag{1.12}$$

is the *growth rate* of the portfolio $\pi(\cdot)$, and

$$\gamma_\pi^*(t) := \frac{1}{2}\Big(\sum_{i=1}^n \pi_i(t) a_{ii}(t) - \sum_{i=1}^n \sum_{j=1}^n \pi_i(t) a_{ij}(t) \pi_j(t) \Big) \tag{1.13}$$

the *excess growth rate* of the portfolio $\pi(\cdot)$. As we shall see in Lemma 3.3 below, for a long-only portfolio this excess growth rate is always non-negative – and is strictly positive for such portfolios that do not concentrate their holdings in just one stock.

Again, the terminology "growth rate" is justified by the a.s. property

$$\lim_{T\to\infty} \frac{1}{T}\Big(\log V^{w,\pi}(T) - \int_0^T \gamma_\pi(t)\, dt \Big) = 0, \tag{1.14}$$

valid under the analogue

$$\lim_{T\to\infty} \left(\frac{\log\log T}{T^2} \int_0^T ||a(t)||\, dt \right) \;=\; 0\,, \qquad \text{a.s.}; \tag{1.15}$$

of condition (1.7). Clearly, this condition is satisfied when all eigenvalues of the covariance matrix process $a(\cdot)$ of (1.3) are *uniformly bounded* away from infinity: i.e., when

$$\xi' a(t) \xi = \xi' \sigma(t) \sigma'(t) \xi \le K \|\xi\|^2, \quad \forall \;\; t \in [0,\infty) \quad \text{and} \quad \xi \in \mathbb{R}^n \tag{1.16}$$

holds almost surely, for some constant $K \in (0,\infty)$. We shall refer to (1.16) as the *uniform boundedness* condition on the volatility structure of $\mathcal{M}$.

Without further comment we shall write $V^\pi(\cdot) \equiv V^{1,\pi}(\cdot)$ for initial wealth $w = \$1$. Let us also note the following analogue of (1.11), namely

$$d \log V^\pi(t) = \gamma_\pi^*(t)\, dt + \sum_{i=1}^n \pi_i(t)\, d\log X_i(t). \tag{1.17}$$

Definition 1.2. We shall use the reverse-order-statistics notation for the weights of a portfolio $\pi(\cdot)$, ranked at time t in decreasing order, from largest down to smallest:

$$\max_{1\le i\le n} \pi_i(t) =: \pi_{(1)}(t) \ge \pi_{(2)}(t) \ge \ldots \ge \pi_{(n-1)}(t) \ge \pi_{(n)}(t) := \min_{1\le i\le n} \pi_i(t). \tag{1.18}$$

For an arbitrary portfolio $\pi(\cdot)$, and with e_i denoting the i^{th} unit vector in $\mathbb{R}^n$, let us introduce the quantities

$$\tau^{\pi}_{ij}(t) := \sum_{\nu=1}^{d} \big(\sigma_{i\nu}(t) - \sigma_{\pi\nu}(t)\big)\big(\sigma_{j\nu}(t) - \sigma_{\pi\nu}(t)\big) \tag{1.19}$$
$$= \big(\pi(t) - e_i\big)' a(t) \big(\pi(t) - e_j\big) = a_{ij}(t) - a_{\pi i}(t) - a_{\pi j}(t) + a_{\pi\pi}(t)$$

for $1 \le i, j \le n$, and set

$$a_{\pi i}(t) := \sum_{j=1}^{n} \pi_j(t) a_{ij}(t), \qquad a_{\pi\pi}(t) := \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_i(t) a_{ij}(t) \pi_j(t) = \sum_{\nu=1}^{d} \big(\sigma_{\pi\nu}(t)\big)^2 . \tag{1.20}$$

It is seen from (1.11) that this last quantity is the variance of the portfolio $\pi(\cdot)$.

We shall call the matrix-valued process $\tau^{\pi}(\cdot) = \big(\tau^{\pi}_{ij}(\cdot)\big)_{1 \le i,j \le n}$ of (1.19) the *process of individual stocks' covariances relative to the portfolio* $\pi(\cdot)$. It satisfies the elementary property

$$\sum_{j=1}^{n} \tau^{\pi}_{ij}(t) \pi_j(t) = 0, \quad i = 1, \cdots, n. \tag{1.21}$$

Trading Strategies: For completeness of exposition and for later use in this survey, let us go briefly beyond portfolios and recall the notion of *trading strategies:* these are allowed to invest in (or borrow from) the money market. Formally, they are $\mathbb{F}-$progressively measurable, $\mathbb{R}^n-$valued processes $h(\cdot) = \big(h_1(\cdot), \cdots h_n(\cdot)\big)'$ that satisfy the integrability condition

$$\sum_{i=1}^{n} \int_0^T \Big(\big|h_i(t)\big| \big|b_i(t) - r(t)\big| + h_i^2(t) a_{ii}(t) \Big)\, dt < \infty, \qquad \text{a.s.}$$

for every $T \in (0, \infty)$. The interpretation is that the real-valued, $\mathcal{F}(t)-$measurable random variable $h_i(t)$ stands for the dollar amount invested by the strategy $h(\cdot)$ at time t in the i^{th} stock. If we denote by $\mathcal{V}^{w,h}(t)$ the wealth at time t corresponding to this strategy $h(\cdot)$ and to an initial capital $w > 0$, then $\mathcal{V}^{w,h}(t) - \sum_{i=1}^{n} h_i(t)$ is the amount invested in the money market, and we have

$$d\mathcal{V}^{w,h}(t) = \Big(\mathcal{V}^{w,h}(t) - \sum_{i=1}^{n} h_i(t)\Big) r(t)\, dt + \sum_{i=1}^{n} h_i(t) \Big(b_i(t)\, dt + \sum_{\nu=1}^{d} \sigma_{i\nu}(t)\, dW_\nu(t) \Big),$$

or equivalently,

$$\frac{\mathcal{V}^{w,h}(t)}{B(t)} = w + \int_0^t \frac{h'(s)}{B(s)} \Big(\big(b(s) - r(s)\mathbf{I}\big)\, ds + \sigma(s)\, dW(s) \Big), \quad 0 \le t < \infty . \tag{1.22}$$

Here $\mathbf{I} = (1, \cdots, 1)'$ is the $n-$dimensional column vector with 1 in all entries. Again without further comment, we shall write $\mathcal{V}^h(\cdot) \equiv \mathcal{V}^{1,h}(\cdot)$ for initial wealth $w = \$1$.

As mentioned already, all quantities $h_i(\cdot),\ 1 \le i \le n$ and $\mathcal{V}^{w,h}(t) - h'(\cdot)\mathbf{I}$ are allowed to take negative values. This possibility opens the door to the notorious *doubling strategies* of martingale theory (e.g. [KS] (1998), Chapter 1). In order to rule these out on a given time-horizon $[0, T]$, we shall confine ourselves here to trading strategies $h(\cdot)$ that satisfy

$$\mathbb{P}\left(\mathcal{V}^{w,h}(t) \ge 0, \ \ \forall\ 0 \le t \le T\right) = 1 . \tag{1.23}$$

Such strategies will be called *admissible* for the initial capital $w > 0$ on the time horizon $[0,T]$; their collection will be denoted $\mathcal{H}(w; T)$, and we shall set $\mathcal{H}(w) := \bigcap_{T>0} \mathcal{H}(w; T)$.

We shall also find useful to look at the collection $\mathcal{H}_+(w; T) \subset \mathcal{H}(w; T)$ of *strongly admissible* strategies, with $\mathbb{P}\big(\mathcal{V}^{w,h}(t) > 0,\ \forall\, 0 \le t \le T\big) = 1$. Similarly, we shall set $\mathcal{H}_+(w) := \bigcap_{T>0} \mathcal{H}_+(w; T)$.

Each portfolio $\pi(\cdot)$ generates, via (1.8), a trading strategy $h(\cdot) \in \mathcal{H}_+(w)$; and we have $\mathcal{V}^{w,h}(\cdot) \equiv V^{w,\pi}(\cdot)$. It is not difficult to see from (1.9) that the trading strategy generated by a portfolio $\pi(\cdot)$ is *self-financing* (see, e.g., Duffie (1992) for a discussion).

2 The Market Portfolio

Suppose we normalize so that each stock has always just one share outstanding; then the stock price $X_i(t)$ can be interpreted as the capitalization of the i^{th} company at time t, and the quantities

$$X(t) := X_1(t) + \ldots + X_n(t) \qquad \text{and} \qquad \mu_i(t) := \frac{X_i(t)}{X(t)}\,, \quad i = 1, \ldots, n \tag{2.1}$$

as the total capitalization of the market and the relative capitalizations of the individual companies, respectively. Clearly $0 < \mu_i(t) < 1, \ \forall\ i = 1, \ldots, n$ and $\sum_{i=1}^n \mu_i(t) = 1$, so we may think of the vector process $\mu(\cdot) = \big(\mu_1(\cdot), \ldots, \mu_n(\cdot)\big)'$ as a portfolio that invests the proportion $\mu_i(t)$ of current wealth in the i^{th} asset at all times. Equivalently, this portfolio holds the same, constant number of shares in all assets, at all times. The resulting wealth process $V^{w,\mu}(\cdot)$ satisfies

$$\frac{dV^{w,\mu}(t)}{V^{w,\mu}(t)} \;=\; \sum_{i=1}^n \mu_i(t)\,\frac{dX_i(t)}{X_i(t)} \;=\; \sum_{i=1}^n \frac{dX_i(t)}{X(t)} \;=\; \frac{dX(t)}{X(t)}\,,$$

in accordance with (2.1) and (1.9). In other words,

$$V^{w,\mu}(\cdot) \equiv \frac{w}{X(0)}\,X(\cdot)\,; \tag{2.2}$$

investing in the portfolio $\mu(\cdot)$ is tantamount to ownership of the entire market, in proportion of course to the initial investment. For this reason, we shall call $\mu(\cdot)$ of (2.1) the *market portfolio*, and the processes μ_i the *market weight processes*.

By analogy with (1.11), we have

$$d\log V^{w,\mu}(t) = \gamma_\mu(t)\,dt + \sum_{\nu=1}^d \sigma_{\mu\nu}(t)\,dW_\nu(t), \qquad V^{w,\mu}(0) = w, \tag{2.3}$$

and comparison of this last equation (2.3) with (1.5) gives the dynamics of the market-weights

$$d\log\mu_i(t) = \big(\gamma_i(t) - \gamma_\mu(t)\big)\,dt + \sum_{\nu=1}^d \big(\sigma_{i\nu}(t) - \sigma_{\mu\nu}(t)\big)\,dW_\nu(t) \tag{2.4}$$

in (2.1) for all stocks $i = 1, \ldots, n$ in the notation of (1.10), (1.12); equivalently,

$$\frac{d\mu_i(t)}{\mu_i(t)} \;=\; \Big(\gamma_i(t) - \gamma_\mu(t) + \frac{1}{2}\tau^\mu_{ii}(t)\Big)\,dt + \sum_{\nu=1}^d \big(\sigma_{i\nu}(t) - \sigma_{\mu\nu}(t)\big)\,dW_\nu(t)\,. \tag{2.5}$$

We are recalling here the quantities

$$\tau^\mu_{ij}(t) := \sum_{\nu=1}^d \big(\sigma_{i\nu}(t) - \sigma_{\mu\nu}(t)\big)\big(\sigma_{j\nu}(t) - \sigma_{\mu\nu}(t)\big) = \frac{d\langle\mu_i, \mu_j\rangle(t)}{\mu_i(t)\mu_j(t)dt}\,, \quad 1 \le i, j \le n \tag{2.6}$$

of (1.19) for the market portfolio $\pi(\cdot) \equiv \mu(\cdot)$, namely, the covariances of the individual stocks relative to the entire market.

Remark 2.1. Coherence: We say that the market model $\mathcal{M}$ of (1.1), (1.2) is *coherent*, if the relative capitalizations of (2.1) satisfy

$$\lim_{T\to\infty} \frac{1}{T}\log\mu_i(T) = 0 \qquad \text{almost surely, for each } i = 1, \cdots, n \tag{2.7}$$

(i.e., if none of the stocks declines too rapidly with respect to the market as a whole). Under the condition (1.15) on the covariance structure, it can be shown that coherence is equivalent to each of the following two conditions:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \big(\gamma_i(t) - \gamma_\mu(t)\big)\, dt = 0 \qquad \text{a.s., for each } \; i = 1, \cdots, n; \tag{2.8}$$

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \big(\gamma_i(t) - \gamma_j(t)\big)\, dt = 0 \quad \text{a.s., for each pair } \; 1 \le i, j \le n. \tag{2.9}$$

See Fernholz (2002), pp. 26-27 for details. □

3 Some Useful Properties

In this section we collect together some useful properties of the relative covariance process in (1.19), for ease of reference in future usage. For any given stock i and portfolio $\pi(\cdot)$, the *relative return process* of the i^{th} stock versus $\pi(\cdot)$ is the process

$$R_i^\pi(t) := \log\left(\frac{X_i(t)}{V^{w,\pi}(t)}\right)\bigg|_{w=X_i(0)} , \qquad 0 \le t < \infty. \tag{3.1}$$

Lemma 3.1. *For any portfolio $\pi(\cdot)$, and for all $1 \le i, j \le n$ and $t \in [0, \infty)$, we have almost surely*

$$\tau_{ij}^\pi(t) = \frac{d}{dt}\langle R_i^\pi, R_j^\pi\rangle(t), \qquad \textit{in particular,} \qquad \tau_{ii}^\pi(t) = \frac{d}{dt}\langle R_i^\pi\rangle(t) \ge 0, \tag{3.2}$$

for the relative covariances of (1.19); and the matrix $\tau^\pi(t) = \big(\tau_{ij}^\pi(t)\big)_{1\le i,j\le n}$ is a.s. nonnegative definite. Furthermore, if the covariance matrix $a(t)$ is positive definite, then the relative covariance matrix $\tau^\pi(t)$ has rank $n-1$, and its null space is spanned by the vector $\pi(t)$, almost surely.

Proof. Comparing (1.5) with (1.11) we get the analogue

$$dR_i^\pi(t) = \big(\gamma_i(t) - \gamma_\pi(t)\big)\, dt + \sum_{\nu=1}^{d} \big(\sigma_{i\nu}(t) - \sigma_{\pi\nu}(t)\big)\, dW_\nu(t),$$

of (2.4), from which the first two claims follow.

Now suppose that $a(t)$ is positive definite. For any $x \in \mathbb{R}^n \setminus \{0\}$ and with $\eta := \sum_{i=1}^n x_i$, we compute from (2.4):

$$x'\tau^\pi(t)x = x'a(t)x - 2\eta x'a(t)\pi(t) + \eta^2\pi'(t)a(t)\pi(t).$$

If $\sum_{i=1}^n x_i = 0$, then $x'\tau^\pi(t)x = x'a(t)x > 0$. If on the other hand $\eta := \sum_{i=1}^n x_i \neq 0$, we consider the vector $y := x/\eta$ that satisfies $\sum_{i=1}^n y_i = 1$, and observe that $\eta^{-2}x'\tau^\pi(t)x$ is equal to

$$y'\tau^\pi(t)y = y'a(t)y - 2y'a(t)\pi(t) + \pi'(t)a(t)\pi(t) = \big(y - \pi(t)\big)'a(t)\big(y - \pi(t)\big),$$

thus zero if and only if $y = \pi(t)$, or equivalently $x = \eta\pi(t)$. □

Lemma 3.2. *For any two portfolios $\pi(\cdot)$, $\rho(\cdot)$ we have*

$$d\,\log\left(\frac{V^\pi(t)}{V^\rho(t)}\right) = \gamma_\pi^*(t)\, dt + \sum_{i=1}^n \pi_i(t)\, d\log\left(\frac{X_i(t)}{V^\rho(t)}\right). \tag{3.3}$$

In particular, we get the dynamics

$$d\,\log\left(\frac{V^\pi(t)}{V^\mu(t)}\right) = \gamma^*_\pi(t)\,dt + \sum_{i=1}^n \pi_i(t)\,d\log\mu_i(t) \tag{3.4}$$
$$= \big(\gamma^*_\pi(t) - \gamma^*_\mu(t)\big)\,dt + \sum_{i=1}^n \big(\pi_i(t) - \mu_i(t)\big)\,d\log\mu_i(t)$$

for the relative return of an arbitrary portfolio $\pi(\cdot)$ *with respect to the market.*

Proof. The equation (3.3) follows from (1.17), and the first equality in (3.4) is the special case of (3.3) with $\rho(\cdot)\equiv\mu(\cdot)$. The second equality in (3.4) follows upon observing from (2.4) that

$$\sum_{i=1}^n \mu_i(t)\,d\log\mu_i(t) = \sum_{i=1}^n \mu_i(t)\big(\gamma_i(t) - \gamma_\mu(t)\big)\,dt = -\gamma^*_\mu(t)\,dt.$$

□

Lemma 3.3. *For any two portfolios* $\pi(\cdot)$, $\rho(\cdot)$ *we have the* numéraire-invariance property

$$\gamma^*_\pi(t) = \frac{1}{2}\Big(\sum_{i=1}^n \pi_i(t)\tau^\rho_{ii}(t) - \sum_{i=1}^n\sum_{j=1}^n \pi_i(t)\pi_j(t)\tau^\rho_{ij}(t)\Big). \tag{3.5}$$

In particular, recalling (1.21), we obtain the representation

$$\gamma^*_\pi(t) = \frac{1}{2}\sum_{i=1}^n \pi_i(t)\tau^\pi_{ii}(t) \tag{3.6}$$

for the excess growth rate, as a weighted average of the individual stocks' variances $\tau^\pi_{ii}(\cdot)$ *relative to the portfolio* $\pi(\cdot)$, *as in (1.19). Whereas from (3.6), (3.2) and Definition 1.1, we get for any long-only portfolio* $\pi(\cdot)$ *the property:*

$$\gamma^*_\pi(t) \ge 0\ . \tag{3.7}$$

Proof. From (1.19) we obtain

$$\sum_{i=1}^n \pi_i(t)\tau^\rho_{ii}(t) = \sum_{i=1}^n \pi_i(t)a_{ii}(t) - 2\sum_{i=1}^n \pi_i(t)a_{\rho i}(t) + a_{\rho\rho}(t)$$

as well as

$$\sum_{i=1}^n\sum_{j=1}^n \pi_i(t)\tau^\rho_{ij}(t)\pi_j(t) = \sum_{i=1}^n\sum_{j=1}^n \pi_i(t)a_{ij}(t)\pi_j(t) - 2\sum_{i=1}^n \pi_i(t)a_{\rho i}(t) + a_{\rho\rho}(t),$$

and (3.5) follows from (1.13). □

For the market portfolio, equation (3.6) becomes

$$\gamma^*_\mu(t) = \frac{1}{2}\sum_{i=1}^n \mu_i(t)\tau^\mu_{ii}(t)\,; \tag{3.8}$$

the summation on the right-hand-side is the average, according to the market weights of individual stocks, of these stocks' variances relative to the market. Thus, (3.8) gives an interpretation of the excess growth rate of the market portfolio, as a measure of the market's "intrinsic" volatility.

Remark 3.1. Note that (3.4), in conjunction with (2.4), (2.5) and the numéraire-invariance property (3.5), implies that for any portfolio $\pi(\cdot)$ we have the *relative return* formula $d\,(V^\pi(t)/V^\mu(t)) = (V^\pi(t)/V^\mu(t))\sum_{i=1}^n(\pi_i(t)/\mu_i(t))\,d\mu_i(t)$, or equivalently, in conjunction with (2.6):

$$d\,\log\left(\frac{V^\pi(t)}{V^\mu(t)}\right) = \sum_{i=1}^n \frac{\pi_i(t)}{\mu_i(t)}\,d\mu_i(t) - \frac{1}{2}\left(\sum_{i=1}^n\sum_{j=1}^n \pi_i(t)\pi_j(t)\tau_{ij}^\mu(t)\right)dt\,. \tag{3.9}$$

$\square$

Lemma 3.4. *Assume that the covariance process $a(\cdot)$ of (1.3) satisfies the following* strong non-degeneracy condition: *there exists a constant $\varepsilon\in(0,\infty)$ such that*

$$\xi' a(t)\xi = \xi'\sigma(t)\sigma'(t)\xi \ge \varepsilon\|\xi\|^2, \quad \forall\ \ t\in[0,\infty) \quad \textit{and} \quad \xi\in\mathbb{R}^n \tag{3.10}$$

holds almost surely (all eigenvalues are bounded away from zero). Then for every portfolio $\pi(\cdot)$ and all $0\le t<\infty$, we have in the notation of (1.18) the inequalities

$$\varepsilon\big(1-\pi_i(t)\big)^2 \le \tau_{ii}^\pi(t), \qquad i=1,\cdots,n\,, \tag{3.11}$$

almost surely. If the portfolio $\pi(\cdot)$ is long-only, we have also

$$\frac{\varepsilon}{2}\big(1-\pi_{(1)}(t)\big) \le \gamma_\pi^*(t). \tag{3.12}$$

Proof. With e_i denoting the i^{th} unit vector in $\mathbb{R}^n$, we have

$$\tau_{ii}^\pi(t) = (\pi(t)-e_i)'a(t)(\pi(t)-e_i) \ge \varepsilon\|\pi(t)-e_i\|^2 = \varepsilon\Big(\big(1-\pi_i(t)\big)^2 + \sum_{j\ne i}\pi_j^2(t)\Big)$$

from (1.19) and (3.10), thus (3.11) follows. Back into (3.6), and with $\pi_i(t)\ge 0$ valid for all $i=1,\cdots,n$, this lower estimate gives

$$\begin{aligned}
\gamma_*^\pi(t) &\ge \frac{\varepsilon}{2}\sum_{i=1}^n \pi_i(t)\Big(\big(1-\pi_i(t)\big)^2 + \sum_{j\ne i}\pi_j^2(t)\Big)\\
&= \frac{\varepsilon}{2}\Big(\sum_{i=1}^n \pi_i(t)\big(1-\pi_i(t)\big)^2 + \sum_{j=1}^n \pi_j^2(t)\big(1-\pi_j(t)\big)\Big)\\
&= \frac{\varepsilon}{2}\sum_{i=1}^n \pi_i(t)\big(1-\pi_i(t)\big) \ge \frac{\varepsilon}{2}\big(1-\pi_{(1)}(t)\big).
\end{aligned}$$

$\square$

Lemma 3.5. *Assume that the uniform boundedness condition (1.16) holds; then for every long-only portfolio $\pi(\cdot)$ and for $0\le t<\infty$, we have in the notation of (1.18) the a.s. inequalities*

$$\tau_{ii}^\pi(t) \le K\big(1-\pi_i(t)\big)\big(2-\pi_i(t)\big), \qquad i=1,\cdots,n \tag{3.13}$$

$$\gamma_\pi^*(t) \le 2K\big(1-\pi_{(1)}(t)\big). \tag{3.14}$$

Proof. By analogy with the previous proof, we get

$$\tau_{ii}^\pi(t) \;\le\; K\Big(\big(1-\pi_i(t)\big)^2 + \sum_{j\ne i}\pi_j^2(t)\Big) \;\le\; K\Big(\big(1-\pi_i(t)\big)^2 + \sum_{j\ne i}\pi_j(t)\Big) \;=\; K(1-\pi_i(t))(2-\pi_i(t))$$

as claimed in (3.13), and bringing this estimate into (3.6) leads to

$$\gamma_*^\pi(t) \;\le\; K \sum_{i=1}^n \pi_i(t)\big(1-\pi_i(t)\big) \;=\; K\Big(\pi_{(1)}(t)\big(1-\pi_{(1)}(t)\big) + \sum_{k=2}^n \pi_{(k)}(t)\big(1-\pi_{(k)}(t)\big)\Big)$$

$$\le\; K\Big(\big(1-\pi_{(1)}(t)\big) + \sum_{k=2}^n \pi_{(k)}(t)\Big) = 2K\big(1-\pi_{(1)}(t)\big).$$

□

Remark 3.2. Portfolio Diversification and Market Volatility as drivers of Growth: Suppose that the market $\mathcal{M}$ of (1.1), (1.2) satisfies the strong-nondegeneracy condition (3.10). Consider a long-only portfolio $\pi(\cdot)$ for which $\pi_{(1)}(t) := \max_{1\le i\le n}\pi_i(t) < 1$ holds for all $t \ge 0$; i.e., which never concentrates its holdings in just one asset. The growth rate of such a portfolio will dominate strictly the average of the individual assets' growth rates: we have almost surely

$$\gamma_\pi(t) - \sum_{i=1}^n \pi_i(t)\,\gamma_i(t) \;=\; \gamma_\pi^*(t) \;\ge\; \frac{\varepsilon}{2}\,\big(1-\pi_{(1)}(t)\big) \;>\; 0\,, \qquad 0 \le t < \infty\,, \tag{3.15}$$

thanks to (1.12) and (3.12). (In particular, if all growth rates $\gamma_i(\cdot) \equiv \gamma(\cdot)$, $i = 1, \cdots, n$ are the same, then the growth rate of such a portfolio will dominate strictly this common growth rate.) The more volatile the market (i.e., the higher the $\varepsilon > 0$ in (3.10)), and the more diversified the portfolio (to wit, the higher the lower-bound $\eta > 0$ in $1-\pi_{(1)}(t) \ge \eta$, $0 \le t < \infty$), the bigger the lower bound of (3.15). In other words, as Fernholz & Shay (1982) were the first to observe: *in the presence of sufficient market volatility, even minimal portfolio diversification can significantly enhance growth.*

To see how significant such an enhancement can be, let us consider **any** *fixed-proportion, long-only portfolio* $\pi(\cdot) \equiv \pi$, for some vector $\pi \in \Delta^n$ with $1-\pi_{(1)} = 1-\max_{1\le i\le n}\pi_i =: \eta > 0$.

(i) From (3.4) and (3.15) we have the a.s. comparisons

$$\frac{1}{T}\,\log\left(\frac{V^\pi(T)}{V^\mu(T)}\right) - \sum_{i=1}^n \frac{\pi_i}{T}\,\log\mu_i(T) \;=\; \frac{1}{T}\int_0^T \gamma_\pi^*(t)\,dt \;\ge\; \frac{\varepsilon\eta}{2} \;>\; 0\,, \qquad \forall\;\; T \in (0,\infty)\,.$$

If the market is *coherent* as in Remark 2.1, we conclude from these comparisons that the wealth corresponding to any such fixed-proportion, long-only portfolio, grows exponentially and at a rate strictly higher than that of the overall market:

$$\liminf_{T\to\infty}\,\frac{1}{T}\,\log\left(\frac{V^\pi(T)}{V^\mu(T)}\right) \;\ge\; \frac{\varepsilon\eta}{2} \;>\; 0\,, \qquad \text{a.s.} \tag{3.16}$$

(ii) Similarly, if the long-term growth rates $\lim_{T\to\infty}(1/T)\log X_i(T) = \gamma_i$ exist a.s. for every $i = 1, \cdots, n$, then (1.17) gives the a.s. comparisons

$$\liminf_{T\to\infty}\,\frac{1}{T}\,\log V^\pi(T) \;\ge\; \sum_{i=1}^n \pi_i\gamma_i + \frac{\varepsilon\eta}{2} \;>\; \sum_{i=1}^n \pi_i\gamma_i\,.$$

4 Portfolio Optimization

We can formulate already some fairly interesting optimization problems.

Problem 4.1 (Quadratic criterion, linear constraint (Markowitz, 1952)). Minimize the portfolio variance $a_{\pi\pi}(t) = \sum_{i=1}^n\sum_{j=1}^n \pi_i(t)a_{ij}(t)\pi_j(t)$, among all portfolios $\pi(\cdot)$ with rate-of-return $b_\pi(t) = \sum_{i=1}^n \pi_i(t)b_i(t) \ge b_0$ greater than, or equal to, a given constant $b_0 \in \mathbb{R}$. □

Problem 4.2 (Quadratic criterion, quadratic constraint). Minimize the portfolio variance

$$a_{\pi\pi}(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i(t) a_{ij}(t) \pi_j(t)$$

among all portfolios $\pi(\cdot)$ with growth rate at least equal to a given constant γ_0, namely:

$$\sum_{i=1}^{n} \pi_i(t) \left(\gamma_i(t) + \frac{1}{2} a_{ii}(t) \right) \geq \gamma_0 + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i(t) a_{ij}(t) \pi_j(t).$$

$\square$

Problem 4.3. *Maximize, over long-only portfolios* $\pi(\cdot)$*, the probability of reaching a given "ceiling"* $\mathfrak{c}$ *before reaching a given "floor"* $\mathfrak{f}$*, with* $0 < \mathfrak{f} < w < \mathfrak{c} < \infty$. More specifically, maximize the probability $\mathbb{P}[\,\mathfrak{T}^{\pi}_{\mathfrak{c}} < \mathfrak{T}^{\pi}_{\mathfrak{f}}\,]$, with the notation $\mathfrak{T}^{\pi}_{\xi} := \inf\{t \geq 0 \,|\, V^{w,\pi}(t) = \xi\}$ for $\xi \in (0, \infty)$.

In the case of constant coëfficients γ_i and a_{ij}, the solution to this problem comes in the following simple form: one looks at the mean-variance, or *signal-to-noise*, ratio

$$\frac{\gamma_\pi}{a_{\pi\pi}} = \frac{\sum_{i=1}^{n} \pi_i (\gamma_i + \frac{1}{2} a_{ii})}{\sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i a_{ij} \pi_j} - \frac{1}{2},$$

and finds a vector $\pi \in \Delta^n$ that maximizes it (Pestien & Sudderth, 1985). $\square$

Problem 4.4. *Minimize, over long-only portfolios* $\pi(\cdot)$*, the expected time* $\mathbb{E}(\mathfrak{T}^{\pi}_{\mathfrak{c}})$ *until a given "ceiling"* $\mathfrak{c} \in (w, \infty)$ *is reached.*

Again with constant coëfficients, it turns out that it is enough to maximize the drift in the equation for $\log V^{w,\pi}(\cdot)$, namely

$$\gamma_\pi = \sum_{i=1}^{n} \pi_i \left(\gamma_i + \tfrac{1}{2} a_{ii} \right) - \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i a_{ij} \pi_j,$$

the portfolio growth rate (Heath, Orey, Pestien & Sudderth, 1987), over vectors $\pi \in \Delta^n$. $\square$

Problem 4.5. *Maximize, over portfolios* $\pi(\cdot)$*, the probability* $\mathbb{P}[\,\mathfrak{T}^{\pi}_{\mathfrak{c}} < T \wedge \mathfrak{T}^{\pi}_{\mathfrak{f}}\,]$ *of reaching a given "ceiling"* $\mathfrak{c}$ *before reaching a given "floor"* $\mathfrak{f}$ *with* $0 < \mathfrak{f} < w < \mathfrak{c} < \infty$*, by a given "deadline"* $T \in (0, \infty)$.

Always with constant coëfficients, suppose there is a vector $\hat{\pi} = (\hat{\pi}_1, \ldots, \hat{\pi}_n)'$ that maximizes *both* the signal-to-noise ratio *and* the variance,

$$\frac{\gamma_\pi}{a_{\pi\pi}} = \frac{\sum_{i=1}^{n} \pi_i (\gamma_i + \frac{1}{2} a_{ii})}{\sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i a_{ij} \pi_j} - \frac{1}{2} \qquad \text{and} \qquad a_{\pi\pi} = \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i a_{ij} \pi_j,$$

respectively, over all vectors $(\pi_1, \cdots, \pi_n)$ that satisfy $\sum_{i=1}^{n} \pi_i = 1$ (as well as $\pi_1 \geq 0, \ldots, \pi_n \geq 0$ if we restrict ourselves to long-only portfolios). Then the resulting constant-proportion portfolio $\hat{\pi}(\cdot) \equiv \hat{\pi}$ is optimal for the above criterion (Sudderth & Weerasinghe, 1989).

This is a big assumption; it is satisfied, for instance, under the (very stringent, and unnatural...) condition that, for some real number $b \leq 0$, we have

$$b_i = \gamma_i + \frac{1}{2} a_{ii} = b\,, \quad \text{for all } \; i = 1, \ldots, n\,.$$

As far as the authors are aware, nobody seems to have solved this problem when such simultaneous maximization is not possible. $\square$

Problem 4.6 (The Growth-Optimal Portfolio). Suppose we can find a portfolio $\widehat{\pi}(\cdot)$ such that, with probability one: *for each* $t \in [0,\infty)$*, the vector* $\widehat{\pi}(t)$ *maximizes the expression*

$$\sum_{i=1}^{n} x_i \gamma_i(t) + \frac{1}{2}\left(\sum_{i=1}^{n} x_i a_{ii}(t) - \sum_{i=1}^{n}\sum_{j=1}^{n} x_i a_{ij}(t) x_j \right) \; = \; x' b(t) - \frac{1}{2} x' a(t)\, x \tag{4.1}$$

over all vectors $(x_1, \cdots, x_n) \in \mathbb{R}^n$ *with* $\sum_{i=1}^n x_i = 1$. In particular, this vector has to satisfy the first-order condition associated with this maximization, namely

$$\big(x - \widehat{\pi}(t)\big)'\big(b(t) - a(t)\widehat{\pi}(t)\big) \; \le \; 0\,, \quad \text{for every vector } \; (x_1, \cdots, x_n) \in \mathbb{R}^n \; \text{ with } \; \sum_{i=1}^{n} x_i = 1\,. \tag{4.2}$$

It is clear then that, for any portfolio $\pi(\cdot)$, we have the a.s. comparison

$$\gamma_\pi(t) \; \le \; \gamma_{\widehat{\pi}}(t)\,, \quad \forall \;\; 0 \le t < \infty \tag{4.3}$$

of growth rates. If (1.15) is satisfied (e.g., if (1.16) holds), then the consequence

$$d \log \left(\frac{V^\pi(t)}{V^{\widehat{\pi}}(t)} \right) \; = \; \big(\gamma_\pi(t) - \gamma_{\widehat{\pi}}(t)\big)\, dt \; + \; \sum_{\nu=1}^{d} \big(\sigma_{\pi\nu}(t) - \sigma_{\widehat{\pi}\nu}(t)\big)\, dW_\nu(t) \tag{4.4}$$

of (1.11) leads to the *growth-optimality property*

$$\limsup_{T\to\infty} \frac{1}{T} \, \log \left(\frac{V^\pi(T)}{V^{\widehat{\pi}}(T)} \right) \; \le \; 0 \quad \text{a.s., for every portfolio } \; \pi(\cdot)\,; \tag{4.5}$$

and if for some $\mathbb{F}-$stopping time T we have $\mathbb{E}\int_0^{\mathrm{T}} ||a(t)||\, dt < \infty$, then (4.3) and (4.4) lead to the *log-optimality property*

$$\mathbb{E}\big(\log V^\pi(\mathrm{T})\big) \; \le \; \mathbb{E}\big(\log V^{\widehat{\pi}}(\mathrm{T})\big)\,, \quad \text{for every portfolio } \;\; \pi(\cdot)\,. \tag{4.6}$$

There is more one can say: denoting by $\widehat{\mathcal{R}}^\pi(\cdot) := V^\pi(\cdot)/V^{\widehat{\pi}}(\cdot)$ the process of (4.4), an application of Itô's rule gives

$$\begin{aligned} \frac{d\,\widehat{\mathcal{R}}^\pi(t)}{\widehat{\mathcal{R}}^\pi(t)} \; &= \; \left[\gamma_\pi(t) - \gamma_{\widehat{\pi}}(t) + \frac{1}{2} \sum_{\nu=1}^{d} \big(\sigma_{\pi\nu}(t) - \sigma_{\widehat{\pi}\nu}(t)\big)^2 \right] dt \; + \; \sum_{\nu=1}^{d} \big(\sigma_{\pi\nu}(t) - \sigma_{\widehat{\pi}\nu}(t)\big)\, dW_\nu(t) \\ &= \; \big(\pi(t) - \widehat{\pi}(t)\big)' \Big[\big(b(t) - a(t)\widehat{\pi}(t)\big)\, dt \; + \; \sigma(t)\, dW(t) \Big]\,. \end{aligned}$$

In conjunction with the first-order condition of (4.2), this semimartingale decomposition shows that $\widehat{\mathcal{R}}^\pi(\cdot)$ is a local supermartingale; because it is positive, this process is therefore a supermartingale, by Fatou's lemma. We obtain the *muméraire property of the growth-optimal portfolio* $\widehat{\pi}(\cdot)$:

$$\widehat{\mathcal{R}}^\pi(\cdot) = V^\pi(\cdot)/V^{\widehat{\pi}}(\cdot) \quad \text{is a supermartingale, for every portfolio } \pi(\cdot)\,. \tag{4.7}$$

□

Chapter II
Diversity & Arbitrage

Roughly speaking, a market is *diverse* if it avoids concentrating all its capital into a single stock, and the *diversity* of a market is a measure of how uniformly the capital is spread among the stocks.

These concepts were introduced in Fernholz (1999); it was shown in Fernholz (2002), Section 3.3, and in Fernholz, Karatzas & Kardaras (2005) that market diversity gives rise to arbitrage. Diversity is a concept that is meaningful for equity markets, but probably not for more general classes of assets. Nevertheless, some of the results in this chapter may be relevant for passive portfolios comprising more general types of assets.

Unlike classical mathematical finance, SPT is not averse to the existence of arbitrage in markets, but rather studies market characteristics that imply the existence of arbitrage. Moreover, it shows that the existence of arbitrage does not preclude the development of option pricing theory or certain types of utility maximization. These and other related ideas are presented in this chapter.

5 Diversity

The notion of diversity for a financial market corresponds to the intuitive (and descriptive) idea, that no single company can ever be allowed to dominate the entire market in terms of relative capitalization. To make this notion precise, let us say that the model $\mathcal{M}$ of (1.1), (1.2) is *diverse* on the time-horizon $[0, T]$, with $T > 0$ a given real number, if there exists a number $\delta \in (0, 1)$ such that the quantities of (2.1) satisfy almost surely

$$\max_{1 \le i \le n} \mu_i(t) =: \mu_{(1)}(t) < 1 - \delta, \quad \forall \ 0 \le t \le T \tag{5.1}$$

in the order-statistics notation of (1.18). In a similar vein, we say that $\mathcal{M}$ is *weakly diverse* on the time-horizon $[0, T]$, if for some $\delta \in (0, 1)$ we have

$$\frac{1}{T} \int_0^T \mu_{(1)}(t) dt < 1 - \delta \, , \quad \text{a.s.} \tag{5.2}$$

We say that $\mathcal{M}$ is *uniformly weakly diverse* on $[T_0, \infty)$, for some real number $T_0 > 0$, if there exists a number $\delta \in (0, 1)$ such that (5.2) holds for every $T \in [T_0, \infty)$.

It follows directly from (3.14) of Lemma 3.5 that, under the uniform boundedness condition (1.16), the model $\mathcal{M}$ of (1.1), (1.2) is diverse (respectively, weakly diverse) on the time-horizon $[0, T]$, if there exists a number $\zeta > 0$ such that

$$\gamma_\mu^*(t) \ge \zeta, \quad \forall \ 0 \le t \le T \qquad \Big(\text{respectively,} \quad \frac{1}{T} \int_0^T \gamma_\mu^*(t) \, dt \ge \zeta \Big) \tag{5.3}$$

holds almost surely. And (3.12) of Lemma 3.4 shows that, under the strong non-degeneracy condition (3.10), the first (respectively, the second) inequality of (5.3) is satisfied if diversity (respectively, weak diversity) holds on the time interval $[0, T]$.

As we shall see in Section 9, diversity can be ensured by a strongly negative rate of growth for the largest stock, resulting in a sufficiently strong repelling drift (e.g., a log-pole-type singularity) away from an appropriate boundary, as well as non-negative growth rates for all the other stocks.

If all the stocks in $\mathcal{M}$ have the same growth rate ($\gamma_i(\cdot) \equiv \gamma(\cdot)$, $\forall \ 1 \le i \le n$) and (1.15) holds, then we have almost surely:

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \gamma_\mu^*(t) \, dt = 0. \tag{5.4}$$

In particular, such an equal-growth-rate market $\mathcal{M}$ cannot be diverse, even weakly, over long time horizons, provided that (3.10) is also satisfied.

Here is a quick argument for these claims: recall that for $X(\cdot) = X_1(\cdot) + \cdots + X_n(\cdot)$ we have

$$\lim_{T \to \infty} \frac{1}{T} \Big(\log X(T) - \int_0^T \gamma_\mu(t) \, dt \Big) = 0, \quad \lim_{T \to \infty} \frac{1}{T} \Big(\log X_i(T) - \int_0^T \gamma(t) \, dt \Big) = 0$$

a.s., from (1.14), (1.6) and $\gamma_i(\cdot) \equiv \gamma(\cdot)$ for all $1 \le i \le n$. But then we have also

$$\lim_{T\to\infty} \frac{1}{T}\Big(\log X_{(1)}(T) - \int_0^T \gamma(t)\, dt \Big) = 0, \quad \text{a.s.}$$

for the biggest stock $X_{(1)}(\cdot) := \max_{1\le i\le n} X_i(\cdot)$, and note the inequalities $X_{(1)}(\cdot) \le X(\cdot) \le nX_{(1)}(\cdot)$. Therefore,

$$\lim_{T\to\infty} \frac{1}{T}\Big(\log X_{(1)}(T) - \log X(T) \Big) = 0, \quad \text{thus} \quad \lim_{T\to\infty} \frac{1}{T}\int_0^T \big(\gamma_\mu(t) - \gamma(t)\big)\, dt = 0\,,$$

almost surely. But $\gamma_\mu(t) = \sum_{i=1}^n \mu_i(t)\gamma(t) + \gamma_\mu^*(t) = \gamma(t) + \gamma_\mu^*(t)$ because of the assumption of equal growth rates, and (5.4) follows. If (3.10) also holds, then (3.12) and (5.4) imply

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \big(1 - \mu_{(1)}(t)\big)\, dt \; = \; 0$$

almost surely, so weak diversity fails on long time horizons: once in a while a single stock dominates the *entire market*, then recedes; sooner or later another stock takes its place as absolutely dominant leader; and so on.

Remark 5.1. If all the stocks in the market $\mathcal{M}$ have *constant* (though not necessarily the same) growth rates, and if (1.16), (3.10) hold, then $\mathcal{M}$ cannot be diverse, even weakly, over long time horizons. □

6 Relative Arbitrage and Its Consequences

The notion of arbitrage is of paramount importance in mathematical finance. We present in this section an allied notion, that of *relative arbitrage*, and explore some of its consequences. In later sections we shall encounter specific, descriptive conditions on market structure, that lead to this form of arbitrage. Relative arbitrage, although discussed here in the context of equity markets, is a concept that remains meaningful for general classes of assets.

Definition 6.1. Given any two portfolios $\pi(\cdot)$, $\rho(\cdot)$ with the same initial capital $V^\pi(0) = V^\rho(0) = 1$, we shall say that $\pi(\cdot)$ represents an *arbitrage opportunity* (respectively, a *strong arbitrage opportunity*) *relative to* $\rho(\cdot)$ over the time-horizon $[0,T]$, with $T > 0$ a given real number, if

$$\mathbb{P}\big(V^\pi(T) \ge V^\rho(T)\big) = 1 \quad \text{and} \quad \mathbb{P}\big(V^\pi(T) > V^\rho(T)\big) > 0 \tag{6.1}$$

(respectively, if $\mathbb{P}(V^\pi(T) > V^\rho(T)) = 1$ holds). We shall say that $\pi(\cdot)$ represents a *superior long-term growth opportunity relative to* $\rho(\cdot)$, if

$$\mathcal{L}^{\pi,\rho} := \liminf_{T\to\infty} \frac{1}{T} \log\left(\frac{V^\pi(T)}{V^\rho(T)}\right) > 0 \quad \text{holds a.s.} \tag{6.2}$$

(Recall here the comparison of (3.16).)

Remark 6.1. The definition of relative arbitrage has historically included the condition that there exist a constant $q = q_{\pi,\rho,T} > 0$ such that

$$\mathbb{P}\big(V^\pi(t) \ge qV^\rho(t), \ \forall\ 0 \le t \le T\big) = 1. \tag{6.3}$$

However, if one can find a portfolio $\pi(\cdot)$ that satisfies the domination properties (6.1) relative to some other portfolio $\rho(\cdot)$, then there exists another portfolio $\widetilde{\pi}(\cdot)$ that satisfies both (6.3) and (6.1) relative to the same $\rho(\cdot)$. The construction involves a strategy of investing a portion $w \in (0,1)$ of the initial capital \$1 in $\pi(\cdot)$, and the remaining proportion $1-w$ in $\rho(\cdot)$. This observation is due to C. Kardaras (2006). □

6.1 Strict Local Martingales

Let us place ourselves now, and for the remainder of this section, within the market model $\mathcal{M}$ of (1.1) under the conditions (1.2). We shall assume further that there exists a *market price of risk* (or "relative risk") $\theta : [0,\infty) \times \Omega \to \mathbb{R}^d$; namely, an $\mathbb{F}-$progressively measurable process with

$$\sigma(t)\theta(t) = b(t) - r(t)\mathbf{I}, \;\; \forall \;\; 0 \le t \le T \qquad \text{and} \qquad \int_0^T \|\theta(t)\|^2 \, dt < \infty \tag{6.4}$$

valid almost surely, for each $T \in (0,\infty)$. (If the volatility matrix $\sigma(\cdot)$ has full rank, namely n, we can take, for instance, $\theta(t) = \sigma'(t)\big(\sigma(t)\sigma'(t)\big)^{-1}[\, b(t) - r(t)\mathbf{I}\,]$ in (6.4).)

In terms of this process $\theta(\cdot)$, we can define the exponential local martingale and supermartingale

$$Z(t) := \exp\Big\{ -\int_0^t \theta'(s)\, dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 \, ds \Big\}, \quad 0 \le t < \infty \tag{6.5}$$

(a martingale, if and only if $\mathbb{E}(Z(T)) = 1\,, \forall\, T \in (0,\infty)$), and the shifted Brownian Motion

$$\widehat{W}(t) := W(t) + \int_0^t \theta(s)\, ds, \qquad 0 \le t < \infty\,. \tag{6.6}$$

Proposition 6.1. A Strict Local Martingale: *Under the assumptions of this subsection, as well as (1.16), suppose that for some real number* $T > 0$ *and for some portfolio* $\rho(\cdot)$ *there exists arbitrage relative to* $\rho(\cdot)$ *on the time-horizon* $[0,T]$. *Then the process* $Z(\cdot)$ *of (6.5) is a* strict *local martingale:* $\mathbb{E}(Z(T)) < 1$.

Proof. Assume, by way of contradiction, that $\mathbb{E}(Z(T)) = 1$. Then from the Girsanov theorem (e.g. [KS], section 3.5) the recipe $\mathbb{Q}_T(A) := \mathbb{E}[Z(T)\, 1_A]\,,\; A \in \mathcal{F}(T)$ defines a probability measure, equivalent to $\mathbb{P}$, under which the process $\widehat{W}(t),\; 0 \le t \le T$ as in (6.6) is Brownian motion.

Under this probability measure $\mathbb{Q}_T$, the discounted stock prices $X_i(\cdot)/B(\cdot),\; i = 1, \cdots, n$ are positive martingales on $[0,T]$, because of

$$d\big(X_i(t)/B(t)\big) \;=\; \big(X_i(t)/B(t)\big) \sum_{\nu=1}^{d} \sigma_{i\nu}(t)\, d\widehat{W}_\nu(t)$$

and of the uniform boundedness condition (1.16). As usual, we express this by saying that $\mathbb{Q}_T$ is then an *equivalent martingale measure* (EMM) for the model, on the given time-horizon $[0,T]$.

More generally, for any portfolio $\pi(\cdot)$, we get then from (6.6) and (1.9):

$$d\big(V^\pi(t)/B(t)\big) \;=\; \big(V^\pi(t)/B(t)\big)\, \pi'(t)\sigma(t)\, d\widehat{W}(t), \qquad V^\pi(0) = 1\,; \tag{6.7}$$

and from (1.16), the discounted wealth process $V^\pi(t)/B(t)\,,\; 0 \le t \le T$ is a positive martingale under $\mathbb{Q}_T$. Thus, the difference $\Delta(t) := (V^\pi(t) - V^\rho(t))/B(t)\,,\; 0 \le t \le T$ is a martingale under $\mathbb{Q}_T$, for any other portfolio $\rho(\cdot)$ with $V^\rho(0) = 1$; consequently, $\mathbb{E}^{\mathbb{Q}_T}\big(\Delta(T)\big) = \Delta(0) = 0$. But this conclusion is inconsistent with (6.1), which mandates $\mathbb{Q}_T\big(\Delta(T) \ge 0\big) = 1$ and $\mathbb{Q}_T\big(\Delta(T) > 0\big) > 0$. $\square$

Now let us consider the *deflated stock-price and wealth processes*

$$\widehat{X}_i(t) := \frac{Z(t)}{B(t)}\, X_i(t), \quad i = 1, \cdots, n\,, \qquad \widehat{X}(t) := \frac{Z(t)}{B(t)}\, X(t) \qquad \text{and} \qquad \widehat{\mathcal{V}}^{w,h}(t) := \frac{Z(t)}{B(t)}\, \mathcal{V}^{w,h}(t) \tag{6.8}$$

for $0 \le t < \infty$, for an arbitrary trading strategy $h(\cdot) \in \mathcal{H}(w)$ admissible for the initial capital $w > 0$. These processes satisfy, respectively, the dynamics

$$d\widehat{X}_i(t) \,= \widehat{X}_i(t) \sum_{\nu=1}^{d} \big(\sigma_{i\nu}(t) - \theta_\nu(t)\big) dW_\nu(t), \quad \widehat{X}_i(0) = x_i\,, \tag{6.9}$$

$$d\widehat{X}(t) = \widehat{X}(t)\sum_{\nu=1}^{d}\big(\sigma_{\mu\nu}(t)-\theta_\nu(t)\big)dW_\nu(t), \quad \widehat{X}(0)=\sum_{i=1}^{n}x_i\,,$$

$$d\widehat{\mathcal{V}}^{w,h}(t) = \left(\frac{Z(t)h'(t)}{B(t)}\sigma(t)-\widehat{\mathcal{V}}^{w,h}(t)\theta'(t)\right)dW(t), \quad \widehat{\mathcal{V}}^{w,h}(0)=w \tag{6.10}$$

in conjunction with (1.1), (1.22) and (6.5). In particular, these processes are non-negative local martingales (and supermartingales) under $\mathbb{P}$.

In other words, the ratio $Z(\cdot)/B(\cdot)$ continues to play its usual rôle as deflator of prices in such a market, even when $Z(\cdot)$ is just a local martingale.

Remark 6.2. Strict Local Martingales Galore: In the setting of Proposition 6.1 with $\rho(\cdot)\equiv\mu(\cdot)$ the market portfolio, it can be shown from (6.9), (6.10) that the deflated stock-price processes $\widehat{X}_i(t)\,,\ 0\le t\le T$ of (6.8) are all *strict* local martingales and (strict) supermartingales:

$$\mathbb{E}\big(\widehat{X}_i(T)\big) < x_i \qquad \text{holds for every } \ i=1,\cdots,n\,. \tag{6.11}$$

□

We shall prove this property below, at the end of the section, based on a more general result: *Under the assumptions of this subsection, suppose that for some real number $T>0$ and for some portfolio $\rho(\cdot)$ there exists arbitrage relative to $\rho(\cdot)$ on the time-horizon $[0,T]$. Then the process $\widehat{V}^{w,\rho}(t):=Z(t)V^{w,\rho}(t)/B(t)\,,\ 0\le t\le T$, defined as in (6.8), is a strict local martingale and a strict supermartingale, namely*

$$\mathbb{E}\big(\widehat{V}^{w,\rho}(T)\big)\ <\ w\,. \tag{6.12}$$

Proposition 6.2. Non-Existence of Equivalent Martingale Measure: *In the context of Proposition 6.1, no Equivalent Martingale Measure can exist for the model $\mathcal{M}$ of (1.1) on $[0,T]$, if the filtration is generated by the driving Brownian Motion $W(\cdot)$: $\mathbb{F}=\mathbb{F}^W$.*

Proof. If $\mathbb{F}=\mathbb{F}^W$, and if the probability measure $\mathbb{Q}$ is equivalent to $\mathbb{P}$ on $\mathcal{F}(T)$, the martingale representation property of the Brownian filtration gives $(d\mathbb{Q}/d\mathbb{P})\big|_{\mathcal{F}(t)}=Z(t)\,,\ 0\le t\le T$ for some process $Z(\cdot)$ of the form (6.5) and some progressively measurable $\theta(\cdot)$ with $\int_0^T||\theta(t)||^2\,dt<\infty$ a.s. Then Itô's rule leads to the extension

$$\frac{d\widehat{X}_i(t)}{\widehat{X}_i(t)} = \left(b_i(t)-r(t)-\sum_{\nu=1}^{d}\sigma_{i\nu}(t)\theta_\nu(t)\right)dt\ +\sum_{\nu=1}^{d}\big(\sigma_{i\nu}(t)-\theta_\nu(t)\big)\,dW_\nu(t)$$

of (6.9) for the deflated stock-prices of (6.8).

But if $\mathbb{Q}$ is an equivalent martingale measure (that is, if all the $X_i(\cdot)/B(\cdot)$'s are $\mathbb{Q}-$martingales on $[0,T]$), then the $\widehat{X}_i(\cdot)$'s are all $\mathbb{P}-$martingales on $[0,T]$, and this leads to the first property $\sigma(t)\,\theta(t)=b(t)-r(t)\mathbf{I}\,,\ 0\le t\le T$ in (6.4). We repeat now the argument of Proposition 6.1 and arrive at a contradiction with (6.1), the existence of relative arbitrage on $[0,T]$. □

6.2 On "Beating the Market"

Let us introduce now the non-increasing, right-continuous function

$$f(t):=\frac{1}{X(0)}\cdot\mathbb{E}\left(\frac{Z(t)}{B(t)}X(t)\right), \qquad 0\le t<\infty\,. \tag{6.13}$$

If relative arbitrage exists on the time-horizon $[0,T]$, with $T>0$ a real number, then we know $f(0)=1>f(T)>0$ from Remark 6.2.

Remark 6.3. With Brownian filtration $\mathbb{F} = \mathbb{F}^W$, with $n = d$ (equal numbers of stocks and driving Brownian motions), and with an invertible volatility matrix $\sigma(\cdot)$, consider the *maximal relative return*

$$\mathfrak{R}(T) := \sup \big\{ r > 0 \,|\, \exists\, h(\cdot) \in \mathcal{H}(1;T) \;\text{ s.t. }\; \big(\mathcal{V}^h(T)/V^\mu(T) \big) \geq r, \text{ a.s.} \big\} \tag{6.14}$$

in excess of the market, that can be obtained by trading strategies over the interval $[0,T]$. It can be shown that this quantity is computed in terms of the function of (6.13), as $\mathfrak{R}(T) = 1/f(T)$. □

Remark 6.4. The shortest time to beat the market by a given amount: Let us place ourselves again under the assumptions of Remark 6.3, and assume that relative arbitrage exists on $[0,T]$ for *every* $T \in (0,\infty)$; see Section 8 for elaboration. For a given "exceedance level" $r > 1$, consider the shortest length of time

$$\mathbf{T}(r) := \inf \big\{ T \in (0,\infty) \,|\, \exists\, h(\cdot) \in \mathcal{H}(1;T) \;\text{ s.t. }\; \big(\mathcal{V}^h(T)/V^\mu(T) \big) \geq r, \text{ a.s.} \big\} \tag{6.15}$$

required to guarantee a return of at least r times the market. It can be shown that this quantity is given by the inverse of the decreasing function $f(\cdot)$ of (6.13) evaluated at $1/r$:

$$\mathbf{T}(r) = \inf \big\{ T \in (0,\infty) \,|\, f(T) \leq 1/r \big\}. \tag{6.16}$$

A detailed argument is presented at the end of subsection 10.1. □

Question: Can the counterparts of (6.14), (6.15) be computed when one is not allowed to use general strategies $h(\cdot) \in \mathcal{H}(1;T)$, but rather long-only portfolios $\pi(\cdot)$?

Remark 6.5. *It is not possible to construct arbitrage relative to the growth-optimal portfolio $\widehat{\pi}(\cdot)$ of Problem 4.6 in Section 4, on any given time-horizon $[0,T]$, with $T > 0$ a real number.* For if such relative arbitrage $\pi(\cdot)$ existed, we would have

$$\mathbb{P}\big[\, \widehat{\mathcal{R}}^\pi(T) \geq 1 \,\big] \,=\, 1 \qquad \text{and} \qquad \mathbb{P}\big[\, \widehat{\mathcal{R}}^\pi(T) > 1 \,\big] \,>\, 0$$

in the notation of (4.7), thus also $\mathbb{E}\big[\, \widehat{\mathcal{R}}^\pi(T) \big] > 1$; but this contradicts the numéraire property (4.7) of the growth-optimal portfolio, which implies $\mathbb{E}\big[\, \widehat{\mathcal{R}}^\pi(T) \big] \leq 1$ for every real number $T > 0$. We owe this observation to C. Kardaras (2006).

In a similar vein, suppose that $u : [0,\infty) \to [0,\infty)$ is a strictly increasing function and that, for some real number $T > 0$ and some portfolio $\rho(\cdot)$, we have the comparison

$$\mathbb{E}\left[\, u\big(V^\pi(T)\big) \,\right] \,\leq\, \mathbb{E}\left[\, u\big(V^\rho(T)\big) \,\right] \quad \text{for every portfolio } \; \pi(\cdot). \tag{6.17}$$

Then it is not possible to construct arbitrage relative to this $\rho(\cdot)$ on the given time-horizon $[0,T]$; for otherwise there would exist a portfolio $\bar{\pi}(\cdot)$ with the properties of (6.1), thus also with the property $\mathbb{E}\left[\, u\big(V^{\bar{\pi}}(T)\big) \,\right] > \mathbb{E}\left[\, u\big(V^\rho(T)\big) \,\right]$ which contradicts (6.17). □

Proof of (6.12). We shall employ the usual notation $V^{w,\rho}(\cdot) = wV^\rho(\cdot)$, $\widehat{V}^{w,\rho}(\cdot)$ for the wealth and the deflated wealth, respectively, of our given portfolio $\rho(\cdot)$ with initial capital $w > 0$. Setting

$$h(\cdot) := V^{w,\rho}(\cdot)\rho(\cdot) \qquad \text{and} \qquad \theta^\rho(\cdot) := \sigma'(\cdot)\rho(\cdot) - \theta(\cdot),$$

the equation (6.10) takes the form $d\widehat{V}^{w,\rho}(t) = \widehat{V}^{w,\rho}(t)\big(\theta^\rho(t)\big)'\, dW(t)$, or equivalently

$$\widehat{V}^{w,\rho}(t) \,=\, w \cdot \widehat{V}^\rho(t) \,=\, w \,\cdot \exp\Big\{ \int_0^t \big(\theta^\rho(s)\big)'\, dW(s) - \frac{1}{2}\int_0^t \|\theta^\rho(s)\|^2\, ds \Big\}, \quad 0 \leq t \leq T. \tag{6.18}$$

On the other hand, introducing the process

$$\widetilde{W}^{(\rho)}(t) \,:=\, W(t) - \int_0^t \theta^\rho(s) ds \,=\, \widehat{W}(t) - \int_0^t \sigma'(s)\rho(s)\, ds, \quad 0 \leq t \leq T, \tag{6.19}$$

we obtain

$$\big(\widehat{V}^{w,\rho}(t)\big)^{-1} \;=\; w^{-1}\cdot \exp\Big\{ -\int_0^t \big(\theta^\rho(s)\big)'\, d\widetilde{W}^{(\rho)}(s) - \frac{1}{2}\int_0^t \|\theta^\rho(s)\|^2 ds\Big\}\,. \tag{6.20}$$

We shall argue (6.12) by contradiction: let us assume that it fails, namely, that $\widehat{V}^{w,\rho}(\cdot)$ of (6.18) is a martingale. From (6.18) and the Girsanov theorem, the process $\widetilde{W}^{(\rho)}(\cdot)$ of (6.19) is then a Brownian motion under the probability measure $\widetilde{\mathbb{P}}^{(\rho)}_T(A) := \mathbb{E}\big(\widehat{V}^{w,\rho}(T)\,1_A\big)/w\,,\ A\in\mathcal{F}(T)$, which is equivalent to $\mathbb{P}$. Then Itô's rule gives

$$d\left(\frac{V^\pi(t)}{V^\rho(t)}\right) \;=\; \left(\frac{V^\pi(t)}{V^\rho(t)}\right)\cdot \sum_{k=1}^n\sum_{\nu=1}^d \big(\pi_k(t)-\rho_k(t)\big)\,\sigma_{k\nu}(t)\, d\widetilde{W}^{(\rho)}_\nu(t) \tag{6.21}$$

for any portfolio $\pi(\cdot)$, in conjunction with (6.7), (6.20) and (6.19); and the ratio $V^\pi(\cdot)/V^\rho(\cdot)$ is seen to be a positive local martingale and supermartingale, under $\widetilde{\mathbb{P}}^{(\rho)}_T$. In particular, we obtain $\widetilde{\mathbb{E}}^{(\rho)}_T\big(V^\pi(T)/V^\rho(T)\big)\le 1$, where $\widetilde{\mathbb{E}}^{(\rho)}_T$ denotes expectation with respect to $\widetilde{\mathbb{P}}^{(\rho)}_T$.

Now consider any portfolio $\pi(\cdot)$ that satisfies the conditions of (6.1) on the time-horizon $[0,T]$, relative to $\rho(\cdot)$; such a portfolio exists by assumption. The first condition in (6.1) gives the comparison $\widetilde{\mathbb{P}}^{(\rho)}_T\,\big(V^\pi(T)\ge V^\rho(T)\big)=1$. In conjunction with the inequality $\widetilde{\mathbb{E}}^{(\rho)}_T\big(V^\pi(T)/V^\rho(T)\big)\le 1$ just proved, we obtain the equality $\widetilde{\mathbb{P}}^{(\rho)}_T\,\big(V^\pi(T)= V^\rho(T)\big)=1$, or equivalently:

$$\mathbb{P}\big(V^\pi(T)= V^\rho(T)\big)=1 \quad \text{for every portfolio } \pi(\cdot) \text{ that satisfies the first condition in (6.1)}\,.$$

But this contradicts the second condition $\mathbb{P}\big(V^\pi(T)> V^\rho(T)\big)>0$ of (6.1). □

Proof of (6.11). From what has already been shown (for (6.12), now applied to the market portfolio), the process $V^{x,\mu}(\cdot)\equiv\widehat{X}(\cdot)=\widehat{X}_1(\cdot)+\cdots+\widehat{X}_n(\cdot)$ is a *strict* local martingale and a *strict* supermartingale. Now each $\widehat{X}_i(\cdot)$ is a positive local (and super-) martingale, so there must exist at least one $j\in\{1,\cdots,n\}$ for which $\widehat{X}_j(\cdot)$ is a strict local martingale and a strict supermartingale.

We shall argue once again by contradiction: suppose that (6.11) fails, to wit, that $\widehat{X}_i(\cdot)$ *is a martingale for some* $i\neq j$. Then (6.21) with $\rho(\cdot)\equiv e_i$ and $\pi(\cdot)\equiv e_j$ gives

$$d\left(\frac{X^j(t)}{X^i(t)}\right) \;=\; \left(\frac{X^j(t)}{X^i(t)}\right)\cdot \sum_{\nu=1}^d \big(\sigma_{j\nu}(t)-\sigma_{i\nu}(t)\big)\, d\widetilde{W}^{(e_i)}_\nu(t)\,,$$

so condition (1.16) implies that $X_j(\cdot)/X_i(\cdot)$ is a $\widetilde{\mathbb{P}}^{(e_i)}_T$–martingale on $[0,T]$. In particular, we get

$$\frac{x_j}{x_i} \;=\widetilde{\mathbb{E}}^{(e_i)}_T\left[\,\frac{X_j(T)}{X_i(T)}\,\right] \;=\; \mathbb{E}\left[\,\frac{Z(T)\,X_j(T)}{B(T)\,x_i}\,\right]\,,$$

which contradicts the strict supermartingale property of $\widehat{X}_j(\cdot)=Z(\cdot)X_j(\cdot)/B(\cdot)$ and proves (6.11). □

7 Diversity leads to Arbitrage

We provide now examples which demonstrate the following principle: *If the model $\mathcal{M}$ of (1.1), (1.2) is weakly diverse over the time-horizon $[0,T]$, and if (3.10) holds, then $\mathcal{M}$ contains strong arbitrage opportunities relative to the market portfolio, at least for sufficiently large real numbers $T>0$.*

The first such examples involve heavily the *diversity-weighted portfolio* $\mu^{(p)}(\cdot)=\big(\mu^{(p)}_1(\cdot),\ldots,\mu^{(p)}_n(\cdot)\big)'$ defined, for some arbitrary but fixed $p\in(0,1)$, in terms of the market portfolio $\mu(\cdot)$ of (2.1) by

$$\mu^{(p)}_i(t) := \frac{\big(\mu_i(t)\big)^p}{\sum_{j=1}^n\big(\mu_j(t)\big)^p}\,, \qquad \forall\ \ i=1,\ldots,n\,. \tag{7.1}$$

Compared to $\mu(\cdot)$, the portfolio $\mu^{(p)}(\cdot)$ in (7.1) decreases the proportion(s) held in the largest stock(s) and increases those placed in the smallest stock(s), while preserving the relative rankings of all stocks; see (7.7) below. It does this in a systematic and 'passive' way, that involves neither parameter estimation nor optimization. The actual performance of this portfolio relative to the S&P 500 index over a 33-year period is discussed in detail in Fernholz (2002), Chapter 7.

We show below that if the model $\mathcal{M}$ is weakly diverse on a time horizon $[0,T]$, with $T>0$ a given real number, then the value process $V^{\mu^{(p)}}(\cdot)$ of the diversity-weighted portfolio in (7.1) satisfies

$$V^{\mu^{(p)}}(T) > V^{\mu}(T)\left(n^{-1/p}\, e^{\,\varepsilon\delta T/2}\right)^{1-p} \tag{7.2}$$

almost surely. In particular,

$$\mathbb{P}\left(V^{\mu^{(p)}}(T) > V^{\mu}(T)\right) = 1, \quad \text{provided that} \quad T \geq \frac{2}{p\varepsilon\delta}\log n\,, \tag{7.3}$$

and $\mu^{(p)}(\cdot)$ is a strong arbitrage opportunity relative to the market $\mu(\cdot)$, in the sense of (6.1). The significance of such a result for practical long-term portfolio management cannot be overstated.

Proof of (7.3). Let us start by introducing the function

$$\mathbf{G}_p(x) := \Big(\sum_{i=1}^{n} x_i^p\Big)^{1/p}, \qquad x \in \Delta_+^n\,, \tag{7.4}$$

which we shall interpret as a "measure of diversity"; see below. An application of Itô's rule to the process $\{\mathbf{G}_p(\mu(t)),\ 0 \leq t < \infty\}$ leads after some computation, and in conjunction with (3.9) and the numéraire-invariance property (3.5), to the expression

$$\log\left(\frac{V^{\mu^{(p)}}(T)}{V^{\mu}(T)}\right) = \log\left(\frac{\mathbf{G}_p(\mu(T))}{\mathbf{G}_p(\mu(0))}\right) + (1-p)\int_0^T \gamma^*_{\mu^{(p)}}(t)\,dt\,, \quad \text{a.s.} \tag{7.5}$$

for the wealth $V^{\mu^{(p)}}(\cdot)$ of the diversity-weighted portfolio $\mu^{(p)}(\cdot)$ of (7.1); see also section 11 below, particularly (11.2) and its proof. One big advantage of the expression (7.5) is that it is free of stochastic integrals, and thus lends itself to pathwise (almost sure) comparisons.

For the function of (7.4), we have the simple bounds

$$1 = \sum_{i=1}^{n}\mu_i(t) \leq \sum_{i=1}^{n}\big(\mu_i(t)\big)^p = \big(\mathbf{G}_p(\mu(t))\big)^p \leq n^{1-p}\,.$$

In other words, the minimum of $\mathbf{G}_p(\mu(t))$ occurs when the entire market is concentrated in one stock ($\mu_j(t)=1$ for some $j \in \{1,\cdots,n\}$), and its maximum when all stocks have the same capitalization ($\mu_1(t)=\cdots=\mu_n(t)=1/n$); this justifies considering the function of (7.4) as a measure of diversity. We deduce the comparison

$$\log\left(\frac{\mathbf{G}_p(\mu(T))}{\mathbf{G}_p(\mu(0))}\right) \geq -\frac{1-p}{p}\log n\,, \quad \text{a.s.} \tag{7.6}$$

which, coupled with (7.5) and (3.7), shows that $V^{\mu^{(p)}}(\cdot)/V^{\mu}(\cdot)$ is bounded from below by the constant $n^{-(1-p)/p}$. In particular, (6.3) is satisfied for $\rho(\cdot) \equiv \mu(\cdot)$ and $\pi(\cdot) \equiv \mu^{(p)}(\cdot)$.

On the other hand, we have already remarked that the biggest weight of the portfolio $\mu^{(p)}(\cdot)$ in (7.1) does not exceed the largest market weight:

$$\mu^{(p)}_{(1)}(t) := \max_{1\leq i\leq n}\mu_i^{(p)}(t) = \frac{\big(\mu_{(1)}(t)\big)^p}{\sum_{k=1}^{n}\big(\mu_{(k)}(t)\big)^p} \leq \mu_{(1)}(t)\,. \tag{7.7}$$

The reverse inequality holds for the smallest weights: $\mu^{(p)}_{(n)}(t) := \min_{1\le i\le n} \mu^{(p)}_i(t) \ge \mu_{(n)}(t)$.

We have assumed that the market is weakly diverse over $[0,T]$, namely, that there is some $0 < \delta < 1$ for which $\int_0^T \big(1-\mu_{(1)}(t)\big)dt > \delta T$ holds almost surely. From (3.12) and (7.7), this implies

$$\int_0^T \gamma^*_{\mu^{(p)}}(t)\,dt \ge \frac{\varepsilon}{2}\int_0^T \left(1-\mu^{(p)}_{(1)}(t)\right)\,dt \ge \frac{\varepsilon}{2}\int_0^T \big(1-\mu_{(1)}(t)\big)\,dt > \frac{\varepsilon}{2}\delta T$$

a.s. In conjunction with (7.6), this leads to (7.2) and (7.3) via

$$\log\left(\frac{V^{\mu^{(p)}}(T)}{V^{\mu}(T)}\right) > (1-p)\left(\frac{\varepsilon T}{2}\delta - \frac{1}{p}\log n\right). \tag{7.8}$$

□

If $\mathcal{M}$ is uniformly weakly diverse and strongly non-degenerate over an interval $[T_0,\infty)$, then (7.8) implies that the market portfolio will lag rather significantly behind the diversity-weighted portfolio over long time horizons. To wit, that (6.2) will hold:

$$\mathcal{L}^{\,\mu^{(p)},\mu} = \liminf_{T\to\infty} \frac{1}{T}\log\left(V^{\mu^{(p)}}(T)/V^{\mu}(T)\right) \ge (1-p)\varepsilon\delta/2 > 0, \qquad \text{a.s.}$$

In Figure 7.1 we see the cumulative changes in the diversity of the U.S. stock market over the period from 1927 to 2004, measured by $\mathbf{G}_p(\cdot)$ with $p = 1/2$. The chart shows the cumulative changes in diversity due to capital gains and losses, rather than absolute diversity, which is affected by changes in market composition and corporate actions. Considering only capital gains and losses has the same effect as adjusting the "divisor" of an equity index. The values used in Figure 7.1 have been normalized so that the average over the whole period is zero. We can observe from the chart that diversity appears to be mean-reverting over the long term, with intermediate trends of 10 to 20 years. The extreme lows for diversity seem to accompany bubbles: the Great Depression, the "nifty fifty" era of the early 1970's, and the "irrational exuberance" period of the late 1990's.

Remark 7.1. (Fernholz, 2002): Under the conditions of this section, consider the portfolio with weights

$$\pi_i(t) = \left(\frac{2-\mu_i(t)}{\mathbf{G}(\mu(t))} - 1\right)\mu_i(t), \quad 1\le i\le n, \quad \text{where} \quad \mathbf{G}(x) := 1 - \frac{1}{2}\sum_{i=1}^n x_i^2$$

for $x \in \Delta^n$. It can be shown that this portfolio leads to arbitrage relative to the market, over sufficiently long time horizons $[0,T]$, namely with $T \ge (2n/\varepsilon\delta^2)\log 2$. In this case, we also have $\pi_i(t) \le 3\mu_i(t)$, for all $t\in[0,T]$, a.s., so, with appropriate initial conditions, there is no risk that this $\pi(\cdot)$ will hold more of a stock than the market holds. □

Remark 7.2. Statistical Arbitrage and Enhanced Indexing. With $p = 1$, the portfolio $\mu^{(p)}(\cdot)$ of (7.1) corresponds to the *market portfolio;* with $p = 0$, it gives the *equally weighted portfolio,* namely, $\varphi_i(\cdot) := \mu^{(0)}_i(\cdot) \equiv 1/n$ for all $i = 1,\cdots,n$.

The market portfolio $\mu(\cdot)$ buys at time $t=0$ the same number of shares in all companies of the market, and holds them until the end $t=T$ of the investing horizon. It represents the quintessential "buy-and-hold" strategy.

The equally weighted portfolio $\varphi(\cdot)$ maintains equal weights in all stocks at all times; it accomplishes this by selling those stocks whose price rises relative to the rest, and by buying stocks whose price falls relative to the others. Because of this built-in aspect of *"buying-low-and-selling-high",* the equally weighted portfolio can be used as a simple prototype for studying systematically the performance of *statistical arbitrage* strategies in equity markets; see Fernholz & Maguire (2006) for details. Of course, implementing such a strategy necessitates very frequent trading and can incur substantial

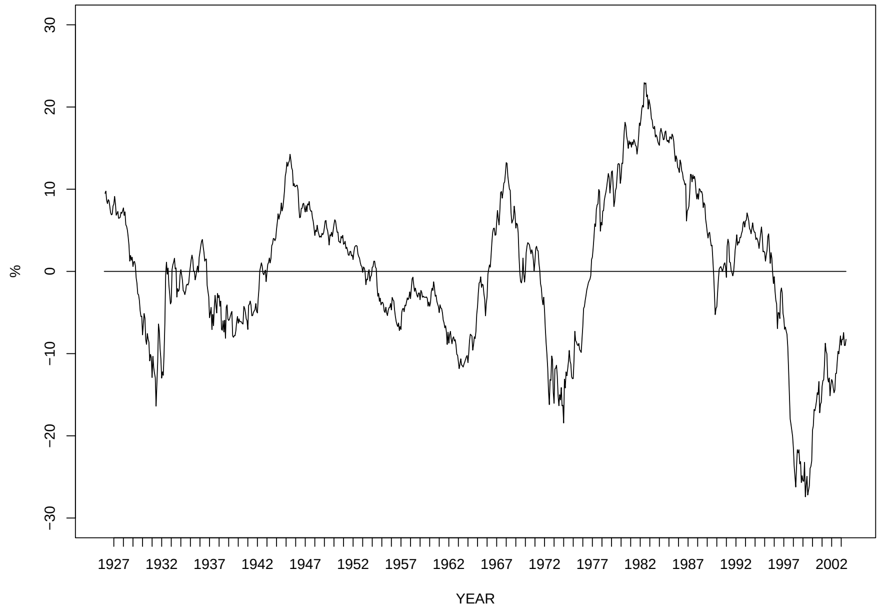

Figure 7.1: Cumulative change in market diversity, 1927–2004.

transaction costs for an investor who is not a broker/dealer. It can also involve considerable risk: whereas the second term on the right-hand side of

$$\log V^{\varphi}(T) \;=\; \frac{1}{n}\log\left(\frac{X_1(T)\cdots X_n(T)}{X_1(0)\cdots X_n(0)}\right) \;+\; \int_0^T \gamma^*_{\varphi}(t)\,dt\,, \tag{7.9}$$

or of

$$\log\left(\frac{V^{\varphi}(T)}{V^{\mu}(T)}\right) \;=\; \frac{1}{n}\log\left(\frac{\mu_1(T)\cdots \mu_n(T)}{\mu_1(0)\cdots \mu_n(0)}\right) \;+\; \int_0^T \gamma^*_{\varphi}(t)\,dt\,, \tag{7.10}$$

is increasing it T, the first terms on the right-hand sides of these expressions can fluctuate quite a bit. These equations are obtained by reading (1.17), (1.13), (3.9) with $\pi_i(\cdot) \equiv \varphi_i(\cdot) \equiv 1/n$ for all $i = 1, \cdots, n$, thus with excess growth rate

$$\gamma^*_{\varphi}(t) \;=\; \frac{1}{2n}\left(\sum_{i=1}^n a_{ii}(t) - \frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n a_{ij}(t)\right)\,. \tag{7.11}$$

The diversity-weighted portfolios $\mu^{(p)}(\cdot)$ of (7.1) with $0 < p < 1$ stand between these two extremes, of capitalization weighting (as in S&P 500) and of equal weighting (as in the Value-Line Index); they try to capture some of the “buy-low/sell-high” characteristics of equal weighting, but without deviating too much from the market capitalizations and without incurring a lot of trading costs or excessive risk. They can be viewed as “enhanced market portfolios” or “enhanced indices”, in this sense. □

8 Mirror Portfolios, Short-Horizon Arbitrage

In the previous section we saw that in weakly diverse markets which satisfy the strict non-degeneracy condition (3.10), one can construct explicitly simple long-only portfolios that lead to strong arbitrages relative to the market over sufficiently long time horizons. The purpose of this section is to demonstrate that, under these same conditions, such arbitrages exist indeed over *arbitrary* time horizons, no matter how small.

For any given portfolio $\pi(\cdot)$ and real number $q \neq 0$, define the *q-mirror image of* $\pi(\cdot)$ *with respect to the market portfolio*, as

$$\widetilde{\pi}^{[q]}(\cdot) := q\pi(\cdot) + (1-q)\mu(\cdot).$$

This is clearly a portfolio; and it is long-only if $\pi(\cdot)$ itself is long-only and $0 < q < 1$. If $q = -1$, we call $\widetilde{\pi}^{[-1]}(\cdot) = 2\mu(\cdot) - \pi(\cdot)$ the "mirror image" of $\pi(\cdot)$ with respect to the market.

By analogy with (1.19), let us define the *relative covariance of* $\pi(\cdot)$ *with respect to the market*, as

$$\tau^{\pi}_{\mu\mu}(t) := \big(\pi(t) - \mu(t)\big)' a(t)\big(\pi(t) - \mu(t)\big), \qquad 0 \leq t \leq T.$$

Remark 8.1. Recall from (1.21) the fact $\tau^{\mu}(t)\mu(t) \equiv 0$, and establish the elementary properties $\tau^{\pi}_{\mu\mu}(t) = \pi'(t)\tau^{\mu}(t)\pi(t) = \tau^{\mu}_{\pi\pi}(t)$ and $\tau^{\mu}_{\widetilde{\pi}^{[q]}\widetilde{\pi}^{[q]}}(t) = q^2\tau^{\mu}_{\pi\pi}(t)$. □

Remark 8.2. The wealth of $\widetilde{\pi}^{[q]}(\cdot)$ relative to the market, can be computed as

$$\log\left(\frac{V^{\widetilde{\pi}^{[q]}}(T)}{V^{\mu}(T)}\right) = q\,\log\left(\frac{V^{\pi}(T)}{V^{\mu}(T)}\right) + \frac{q(1-q)}{2}\int_0^T \tau^{\mu}_{\pi\pi}(t)\,dt.$$

Indeed, let us write the second equality in (3.4) with $\pi(\cdot)$ replaced by $\widetilde{\pi}^{[q]}(\cdot)$, and recall $\widetilde{\pi}^{[q]} - \mu = q(\pi - \mu)$. From the resulting expression, let us subtract the second equality in (3.4), now multiplied by q; the result is

$$\frac{d}{dt}\left(\log\frac{V^{\widetilde{\pi}^{[q]}}(t)}{V^{\mu}(t)} - q\,\log\frac{V^{\pi}(t)}{V^{\mu}(t)}\right) = (q-1)\gamma^*_{\mu}(t) + \big(\gamma^*_{\widetilde{\pi}^{[q]}}(t) - q\gamma^*_{\mu}(t)\big).$$

But from the equalities of Remark 8.1 and Lemma 3.3, we obtain

$$\begin{aligned}
2\big(\gamma^*_{\widetilde{\pi}^{[q]}}(t) - q\gamma^*_{\pi}(t)\big) &= \sum_{i=1}^{n}\big(\widetilde{\pi}^{[q]}(t) - q\pi_i(t)\big)\tau^{\mu}_{ii}(t) - \tau^{\mu}_{\widetilde{\pi}^{[q]}\widetilde{\pi}^{[q]}}(t) + q\tau^{\mu}_{\pi\pi}(t) \\
&= (1-q)\sum_{i=1}^{n}\mu_i(t)\tau^{\mu}_{ii}(t) + q\tau^{\mu}_{\pi\pi}(t) - q^2\tau^{\mu}_{\pi\pi}(t) = (1-q)\big(2\gamma^*_{\mu}(t) + q\tau^{\mu}_{\pi\pi}(t)\big).
\end{aligned}$$

The desired equality now follows. □

Remark 8.3. Suppose that the portfolio $\pi(\cdot)$ satisfies

$$\mathbb{P}\big(V^{\pi}(T)/V^{\mu}(T) \geq \beta\big) = 1 \qquad \text{or} \qquad \mathbb{P}\big(V^{\pi}(T)/V^{\mu}(T) \leq 1/\beta\big) = 1$$

and

$$\mathbb{P}\Big(\int_0^T \tau^{\mu}_{\pi\pi}(t)\,dt \geq \eta\Big) = 1$$

for some real numbers $T > 0$, $\eta > 0$ and $0 < \beta < 1$. Then there exists another portfolio $\widehat{\pi}(\cdot)$ with $\mathbb{P}\big(V^{\widehat{\pi}}(T) < V^{\mu}(T)\big) = 1$.

To see this, suppose first that we have $\mathbb{P}\big(V^{\pi}(T)/V^{\mu}(T) \leq 1/\beta\big) = 1$; then we can just take $\widehat{\pi}(\cdot) \equiv \widetilde{\pi}^{[q]}(\cdot)$ with $q > 1 + (2/\eta)\log(1/\beta)$, for then Remark 8.2 gives

$$\log\left(\frac{V^{\widetilde{\pi}^{[q]}}(T)}{V^{\mu}(T)}\right) \leq q\left(\log\big(1/\beta\big) + \frac{1-q}{2}\eta\right) < 0, \qquad \text{a.s.}$$

If, on the other hand, $\mathbb{P}\big(V^{\pi}(T)/V^{\mu}(T) \geq \beta\big) = 1$ holds, then similar reasoning shows that it suffices to take $\widehat{\pi}(\cdot) \equiv \widetilde{\pi}^{[q]}(\cdot)$ with $q \in \big(0, 1 - (2/\eta)\log(1/\beta)\big)$. □

8.1 A "Seed" Portfolio

Now let us consider $\pi = e_1 = (1, 0, \cdots, 0)'$ and the market portfolio $\mu(\cdot)$; we shall fix a real number $q > 1$ in a moment, and define the portfolio

$$\widehat{\pi}(t) := \widetilde{\pi}^{[q]}(t) = q e_1 + (1-q)\mu(t), \qquad 0 \le t < \infty \tag{8.1}$$

which takes a long position in the first stock and a short position in the market. In particular, $\widehat{\pi}_1(t) = q + (1-q)\mu_1(t)$ and $\widehat{\pi}_i(t) = (1-q)\mu_i(t)$ for $i = 2, \cdots, n$. Then we have

$$\log\left(\frac{V^{\widehat{\pi}}(T)}{V^{\mu}(T)}\right) \;=\; q\,\log\left(\frac{\mu_1(T)}{\mu_1(0)}\right) - \frac{q(q-1)}{2}\int_0^T \tau_{11}^{\mu}(t)dt \tag{8.2}$$

from Remark 8.2. But taking $\beta := \mu_1(0)$ we have $(\mu_1(T)/\mu_1(0)) \le 1/\beta$; and if the market is weakly diverse on $[0,T]$ and satisfies the strict non-degeneracy condition (3.10), we obtain from (3.11) and the Cauchy-Schwarz inequality

$$\int_0^T \tau_{11}^{\mu}(t)dt \ge \varepsilon \int_0^T \big(1-\mu_{(1)}\big)^2 dt > \varepsilon\delta^2 T =: \eta\,. \tag{8.3}$$

Recalling Remark 8.3, we see that *the market portfolio represents then a strong arbitrage opportunity with respect to the portfolio $\widehat{\pi}(\cdot)$ of (8.1)*, provided that for any given real number $T > 0$ we select

$$q > q(T) := 1 + (2/\varepsilon\delta^2 T)\log\big(1/\mu_1(0)\big). \tag{8.4}$$

The portfolio $\widehat{\pi}(\cdot)$ of (8.1) can be used as a "seed", to create *long-only* portfolios that outperform the market portfolio $\mu(\cdot)$, *over any time-horizon* $[0,T]$ *with given real number* $T > 0$. The idea is to immerse $\widehat{\pi}(\cdot)$ in a sea of market portfolio, swamping the short positions while retaining the essential portfolio characteristics. Crucial in these constructions is following the a.s. comparison, a consequence of (8.2):

$$V^{\widehat{\pi}}(t) \le \left(\frac{\mu_1(t)}{\mu_1(0)}\right)^q V^{\mu}(t)\,, \qquad 0 \le t < \infty\,. \tag{8.5}$$

8.2 Relative Arbitrage on Arbitrary Time Horizons

To implement this idea, consider a strategy $h(\cdot)$ that, at time $t=0$, invests $q/(\mu_1(0))^q$ dollars in the market portfolio, goes one dollar short in the portfolio $\widehat{\pi}(\cdot)$ of (8.1), and makes no change thereafter. The number $q > 1$ is chosen again as in (8.4). The wealth generated by this strategy, with initial capital $z := q/(\mu_1(0))^q - 1 > 0$, is

$$\mathcal{V}^{z,h}(t) = \frac{qV^{\mu}(t)}{(\mu_1(0))^q} - V^{\widehat{\pi}}(t) \ge \frac{V^{\mu}(t)}{(\mu_1(0))^q}\big(q - (\mu_1(t))^q\,\big) > 0, \qquad 0 \le t < \infty, \tag{8.6}$$

thanks to (8.5) and $q > 1 > (\mu_1(t))^q$. This process $\mathcal{V}^{z,h}(\cdot)$ coincides with the wealth $V^{z,\eta}(\cdot)$ generated by a portfolio $\eta(\cdot)$ with weights

$$\eta_i(t) = \frac{1}{\mathcal{V}^{z,h}(t)}\left(\frac{q\mu_i(t)}{(\mu_1(0))^q}V^{\mu}(t) - \widehat{\pi}_i(t)V^{\widehat{\pi}}(t)\right), \qquad i = 1, \cdots, n \tag{8.7}$$

that satisfy $\sum_{i=1}^n \eta_i(t) = 1$. Now we have $\widehat{\pi}_i(t) = -(q-1)\mu_i(t) < 0$ for $i = 2, \cdots, n$, so the quantities $\eta_2(\cdot), \ldots, \eta_n(\cdot)$ are strictly positive. To check that $\eta(\cdot)$ is a long-only portfolio, we have to verify $\eta_1(t) \ge 0$; but the dollar amount invested by $\eta(\cdot)$ in the first stock at time t, namely

$$\frac{q\mu_1(t)}{(\mu_1(0))^q}\,V^{\mu}(t) - \big[q - (q-1)\mu_1(t)\big]V^{\widehat{\pi}}(t)$$

dominates $\frac{q\mu_1(t)}{(\mu_1(0))^q} V^\mu(t) - \big[q-(q-1)\mu_1(t)\big]\left(\frac{\mu_1(t)}{\mu_1(0)}\right)^q V^\mu(t)$, or equivalently

$$\frac{V^\mu(t)\mu_1(t)}{(\mu_1(0))^q}\Big((q-1)(\mu_1(t))^q + q\big[1-(\mu_1(t))^{q-1}\big]\Big) > 0\,,$$

again thanks to (8.5) and $q > 1 > (\mu_1(t))^{q-1}$. Thus $\eta(\cdot)$ is indeed a long-only portfolio.

On the other hand, $\eta(\cdot)$ *outperforms at $t = T$ a market portfolio that starts with the same initial capital* at $t = 0$; this is because $\eta(\cdot)$ is long in the market $\mu(\cdot)$ and short in the portfolio $\widehat{\pi}(\cdot)$, which underperforms the market at $t = T$. Indeed, from Remark 8.3 we have

$$V^{z,\eta}(T) \;=\; \frac{q}{(\mu_1(0))^q} V^\mu(T) - V^{\widehat{\pi}}(T) > zV^\mu(T) = V^{z,\mu}(T)\,, \quad \text{a.s.}$$

Note, however, that as $T \downarrow 0$, the initial capital $z(T) = q(T)/(\mu_1(0))^{q(T)} - 1$ required to do all of this, increases without bound: It may take a huge amount of initial investment to realize the extra basis point's worth of relative arbitrage over a short time horizon — confirming of course, if confirmation is needed, that *time is money...*

9 A Diverse Market Model

The careful reader might have been wondering whether the theory we have developed so far may turn out to be vacuous. *Do there exist market models of the form* (1.1), (1.2) *that are diverse, at least weakly?* This is of course a very legitimate question.

Let us mention then, rather briefly, an example of such a market model $\mathcal{M}$ which is diverse over any given time horizon [0,T] with real $T > 0$. For the details of this construction we refer to [FKK] (2005).

With given $\delta \in (1/2, 1)$, equal numbers of stocks and driving Brownian motions (that is, $d = n$), constant volatility matrix σ that satisfies (3.10), and non-negative numbers $g_1, \ldots, g_n$, we take a model

$$d \log X_i(t) = \gamma_i(t)\, dt + \sum_{\nu=1}^{n} \sigma_{i\nu}\, dW_\nu(t), \qquad 0 \le t \le T \tag{9.1}$$

in the form (1.5) for the vector $\mathfrak{X}(\cdot) = \big(X_1(\cdot), \cdots, X_n(\cdot)\big)'$ of stock prices. With the usual notation $X(t) = \sum_{j=1}^n X_j(t)$, its growth rates are specified as

$$\gamma_i(t) := g_i 1_{\mathcal{Q}_i^c}(\mathfrak{X}(t)) - \frac{M}{\delta}\,\frac{1_{\mathcal{Q}_i}(\mathfrak{X}(t))}{\log\big((1-\delta)X(t)/X_i(t)\big)}. \tag{9.2}$$

In other words, $\gamma_i(t) = g_i \ge 0$ if $\mathfrak{X}(t) \notin \mathcal{Q}_i$ (the i^{th} stock does not have the largest capitalization); and

$$\gamma_i(t) = -\frac{M}{\delta}\,\frac{1}{\log\big((1-\delta)/\mu_i(t)\big)}, \qquad \text{if} \quad \mathfrak{X}(t) \in \mathcal{Q}_i \tag{9.3}$$

(the i^{th} stock *does* have the largest capitalization). We are setting here

$$\mathcal{Q}_1 := \Big\{x \in (0,\infty)^n \,\big|\, x_1 \ge \max_{2\le j\le n} x_j\Big\}, \quad \mathcal{Q}_n := \Big\{x \in (0,\infty)^n \,\big|\, x_n > \max_{1\le j\le m-1} x_j\Big\},$$
$$\mathcal{Q}_i := \Big\{x \in (0,\infty)^n \,\big|\, x_i > \max_{1\le j\le i-1} x_j,\ x_i \ge \max_{i+1\le j\le n} x_j\Big\} \quad \text{for} \ \ i = 2, \ldots, n-1.$$

With this specification (9.2), (9.3), all stocks but the largest behave like geometric Brownian motions (with growth rates $g_i \ge 0$ and variances $a_{ii} = \sum_{\nu=1}^n \sigma_{i\nu}^2$), whereas the log-price of the largest stock is subjected to a log-pole-type singularity in its drift, away from an appropriate right boundary.

One can then show that the resulting system of stochastic differential equations has a unique, strong solution (so the filtration $\mathbb{F}$ is now the one generated by the driving $n-$dimensional Brownian motion), and that the diversity requirement (5.1) is satisfied on any given time horizon. Such models can be modified appropriately, to create ones that are weakly diverse but not diverse; see [FK] (2005) for details.

Slightly more generally, in order to guarantee diversity it is enough to require

$$\min_{2\le k\le n} \gamma_{(k)}(t) \ge 0 \ge \gamma_{(1)}(t), \qquad \min_{2\le k\le n} \gamma_{(k)}(t) - \gamma_{(1)}(t) + \frac{\varepsilon}{2} \ge \frac{M}{\delta} F(Q(t)),$$

where $Q(t) := \log\big((1-\delta)/\mu_{(1)}(t)\big)$.

Here the function $F : (0,\infty) \to (0,\infty)$ is taken to be continuous, and such that the associated scale function

$$U(x) := \int_1^x \exp\Big\{ - \int_1^y F(z)\, dz \Big\} dy, \quad x \in (0,\infty) \qquad \text{satisfies} \quad U(0+) = -\infty;$$

for instance, we have $U(x) = \log x$ when $F(x) = 1/x$ as above. Under these conditions, it can then be shown that the process $Q(\cdot)$ satisfies $\int_0^T (Q(t))^{-2} dt < \infty$ a.s., and this leads to the a.s. square-integrability

$$\sum_{i=1}^n \int_0^T (b_i(t))^2 dt < \infty \tag{9.4}$$

of the induced rates of return of the individual stocks

$$b_i(t) \;=\; \frac{1}{2} a_{ii} + g_i 1_{\mathcal{Q}_i^c}(\mathfrak{X}(t)) - \frac{M}{\delta} \frac{1_{\mathcal{Q}_i}(\mathfrak{X}(t))}{\log\big((1-\delta)X(t)/X_i(t)\big)}\,, \qquad i = 1, \cdots, n.$$

The square-integrability property (9.4) is, of course, crucial: it guarantees that the market-price-of-risk process $\theta(\cdot) := \sigma^{-1} b(\cdot)$ is square-integrable a.s., exactly as posited in (6.4), so the exponential local martingale $Z(\cdot)$ of (6.5) is well defined (we are assuming $r(\cdot) \equiv 0$ in all this). Thus the results of Propositions 6.1, 6.2 and Remark 6.2 are applicable to this model.

For additional examples, and for an interesting probabilistic construction of diverse markets that leads to arbitrage, see Osterrieder & Rheinländer (2006).

10 Hedging and Optimization without EMM

Let us broach now the issue of hedging contingent claims in a market such as that of subsection 6.1, and over a time-horizon $[0, T]$ with a real number $T > 0$ satisfying (6.1).

Consider first a *European contingent claim*, that is, an $\mathcal{F}(T)-$measurable random variable $Y : \Omega \to [0,\infty)$ with

$$0 < y := \mathbb{E}\big(Y Z(T)/B(T)\big) < \infty \tag{10.1}$$

in the notation of (6.5). From the point of view of the seller of the contingent claim (e.g., stock option), this random amount represents a liability that has to be covered with the right amount of initial funds at time $t = 0$ and the right trading strategy during the interval $[0, T]$, so that at the end of the time-horizon (time $t = T$) the initial funds have grown enough, to cover the liability without risk. Thus, the seller is interested in the so-called *upper hedging price*

$$\mathcal{U}^Y(T) := \inf\big\{ w > 0 \,|\, \exists\, h(\cdot) \in \mathcal{H}(w; T) \;\text{ s.t. } \mathcal{V}^{w,h}(T) \ge Y, \text{ a.s.}\big\}, \tag{10.2}$$

the smallest amount of initial capital that makes such riskless hedging possible.

The standard theory of mathematical finance assumes that $\mathfrak{M}$, the set of equivalent martingale measures for the model $\mathcal{M}$, is non-empty; then shows that $\mathcal{U}^Y(T)$ can be computed as

$$\mathcal{U}^Y(T) \,=\, \sup_{\mathbb{Q}\in\mathfrak{M}} \mathbb{E}^{\mathbb{Q}}\big(Y/B(T)\big), \tag{10.3}$$

the supremum of the claim's discounted expected values over this set of probability measures. In our context no EMM exists (that is, $\mathfrak{M} = \emptyset$), so the approach breaks down and the problem seems hopeless.

Not quite, though: there is still a long way one can go, simply by utilizing the availability of the strict local martingale $Z(\cdot)$ (and of the associated "deflator" $Z(\cdot)/B(\cdot)$), as well as the properties (6.9), (6.10) of the processes in (6.8). For instance, if the set on the right-hand side of (10.2) is not empty, then for any $w > 0$ in this set and for any $h(\cdot) \in \mathcal{H}(w;T)$, the local martingale $\widehat{\mathcal{V}}^{w,h}(\cdot)$ of (6.8) is non-negative, thus a supermartingale. This gives

$$w \geq \mathbb{E}\big(\mathcal{V}^{w,h}(T)Z(T)/B(T)\big) \geq \mathbb{E}\big(YZ(T)/B(T)\big) = y,$$

and because $w > 0$ is arbitrary we deduce $\mathcal{U}^Y(T) \geq y$. This inequality holds trivially if the set on the right-hand side of (10.2) is empty, since then we have $\mathcal{U}^Y(T) = \infty$.

10.1 Completeness without EMM

To obtain the reverse inequality we shall assume that $n = d$, i.e., that we have exactly as many sources of randomness as there are stocks in the market $\mathcal{M}$, and that the filtration $\mathbb{F}$ is generated by the driving Brownian Motion $W(\cdot)$ in (1.1): $\mathbb{F} = \mathbb{F}^W$.

With these assumptions, one can represent the non-negative martingale

$$M(t) \,:=\, \mathbb{E}\big(YZ(T)/B(T)\,|\,\mathcal{F}(t)\big)\,, \qquad 0 \leq t \leq T$$

as a stochastic integral

$$M(t) = y + \int_0^t \psi'(s)dW(s)\,, \qquad 0 \leq t \leq T \tag{10.4}$$

for some progressively measurable and a.s. square-integrable process $\psi : [0,T]\times\Omega \to \mathbb{R}^d$ and with the notation of (10.1). Setting

$$V_*(\cdot) \,:=\, M(\cdot)B(\cdot)/Z(\cdot) \qquad \text{and} \qquad h_*(\cdot) \,:=\, \big(B(\cdot)/Z(\cdot)\big)\, a^{-1}(\cdot)\,\sigma(\cdot)\big(\psi(\cdot) + M(\cdot)\theta(\cdot)\big)\,,$$

then comparing (6.10) with (10.4), we observe that $V_*(0) = y$, $V_*(T) = Y$ and $V_*(\cdot) \equiv \mathcal{V}^{y,h_*}(\cdot) \geq 0$ hold almost surely.

Therefore, the trading strategy $h_*(\cdot)$ is in $\mathcal{H}(y;T)$ and satisfies the *exact replication property* $\mathcal{V}^{y,h_*}(T) = Y$ a.s. This implies that y belongs to the set on the right-hand side of (10.2), and so $y \geq \mathcal{U}^Y(T)$. But we have already established the reverse inequality, actually in much greater generality, so recalling (10.1) we get the *Black-Scholes-type formula*

$$\mathcal{U}^Y(T) = \mathbb{E}\big(YZ(T)/B(T)\big) \tag{10.5}$$

for the upper hedging price of (10.2), under the assumptions of the first paragraph in this subsection.

In particular, we see that a market $\mathcal{M}$ which is weakly diverse – hence without an equivalent probability measure under which discounted stock prices are (at least local) martingales – can nevertheless be *complete*.

Similar observations have been made by Lowenstein & Willard (2000.a,b) and by Platen (2002, 2006).

Remark 10.1. Put-Call Parity. In the context of this subsection, suppose $L_1(\cdot)$ and $L_2(\cdot)$ are positive, continuous and adapted processes, representing the values of two different financial instruments in the market. For instance, $L_1(\cdot) = V^{w_1,\pi_1}(\cdot)$ and $L_2(\cdot) = V^{w_2,\pi_2}(\cdot)$ for two different portfolios $\pi_1(\cdot)$, $\pi_2(\cdot)$ and real numbers $w_1 > 0$, $w_2 > 0$. Consider the contingent claims

$$Y_1 := \big(L_1(T) - L_2(T)\big)^+ , \qquad Y_2 := \big(L_2(T) - L_1(T)\big)^+ .$$

According to (10.5), the quantity $\mathcal{U}_1 = \mathbb{E}\,[\,Z(T)Y_1/B(T)\,]$ is the upper hedging price at $t = 0$ of a contingent claim that confers to its holder the right, thought not the obligation, to exchange instrument 2 for instrument 1 at time $t = T$; ditto for $\mathcal{U}_2 = \mathbb{E}\,[\,Z(T)Y_2/B(T)\,]$, with the rôles of instruments 1 and 2 interchanged. Of course,

$$\mathcal{U}_1 - \mathcal{U}_2 = \mathbb{E}\,\big[\,Z(T)\,\big(L_1(T) - L_2(T)\big)/B(T)\,\big]\,;$$

and we say that the two instruments are *in Put-Call Parity,* if $\mathcal{U}_1 - \mathcal{U}_2 = L_1(0) - L_2(0)$. This will be the case, for instance, if $Z(\cdot)\big(L_1(\cdot) - L_2(\cdot)\big)/B(\cdot)$ is a martingale.

Put-Call Parity can fail, when relative arbitrage of the type (6.1) exists. For example, take $L_1(\cdot) \equiv V^{\pi}(\cdot)$, $L_2(\cdot) \equiv V^{\rho}(\cdot)$ and observe that (6.1) leads to

$$\mathcal{U}_1 - \mathcal{U}_2 = \mathbb{E}\,\big[\,Z(T)\,\big(V^{\pi}(T) - V^{\rho}(T)\big)/B(T)\,\big] > 0 = V^{\pi}(0) - V^{\rho}(0)\,.$$

□

Proof of (6.16). We can provide now a proof for the claim (6.16) in Remark 6.4. Let us denote by $\mathbf{T}$ the right-hand side of this equation, and note that the inequality $\mathbf{T} \le \mathbf{T}(r)$ is automatically satisfied if the set in (6.15) is empty (its infimum is then $+\infty$); if the set in (6.15) is not empty, pick any element $T \in (0, \infty)$ and an arbitrary trading strategy $h(\cdot) \in \mathcal{H}(1;T)$ that satisfies $\mathcal{V}^h(T) \ge r \cdot V^{\mu}(T)$ a.s. The supermartingale property of $Z(\cdot)\mathcal{V}^h(\cdot)/B(\cdot)$ gives then

$$1 \ge \mathbb{E}\big[\,Z(T)\mathcal{V}^h(T)/B(T)\,\big] \ge r \cdot \mathbb{E}\big[\,Z(T)V^{\mu}(T)/B(T)\,\big] = r \cdot f(T)\,,$$

which means that this $T \in (0, \infty)$ belongs to the set of (6.16); thus the inequality $\mathbf{T} \le \mathbf{T}(r)$ holds again.

For the reverse inequality, consider the number $y := f(\mathbf{T})$ and observe $0 < y \le 1/r$ (the right-continuity of $f(\cdot)$). From what we just proved, there exists a trading strategy $h_*(\cdot) \in \mathcal{H}(1;\mathbf{T})$ with which the contingent claim $Y := X(\mathbf{T})/X(0)$ can be *replicated exactly* at time $t = \mathbf{T}$, in the sense $y\,\mathcal{V}^{h_*}(\mathbf{T}) = Y$ a.s., since $\mathbb{E}\big[\,Z(\mathbf{T})Y/B(\mathbf{T})\,\big] = y$. Therefore,

$$(1/r) \cdot \mathcal{V}^{h_*}(\mathbf{T}) \ge y \cdot \mathcal{V}^{h_*}(\mathbf{T}) = Y = X(\mathbf{T})/X(0) = V^{\mu}(\mathbf{T}) \quad \text{holds a.s.,}$$

and this means that $\mathbf{T}$ belongs to the set of (6.16); thus the inequality $\mathbf{T} \ge \mathbf{T}(r)$ holds as well. □

10.2 Ramifications and Open Problems

Example 10.1. A European Call Option. Consider the contingent claim $Y = \big(X_1(T) - q\big)^+$: this is a European call option on the first stock, with strike $q \in (0, \infty)$ and expiration $T \in (0, \infty)$. Let us assume also that the interest-rate process $r(\cdot)$ is bounded away from zero, namely that $\mathbb{P}[\,r(t) \ge r,\ \forall\, t \ge 0\,] = 1$ holds for some $r > 0$, and that the market $\mathcal{M}$ is weakly diverse on all sufficiently large time horizons $T \in (0, \infty)$.

Then for the hedging price $\mathcal{U}^Y(T)$ of this contingent claim we have from Remark 6.2, (10.5), Jensen's inequality, and $\mathbb{E}(Z(T)) < 1$:

$$\begin{aligned}
X_1(0) &> \mathbb{E}\big(Z(T)X_1(T)/B(T)\big) \ge \mathbb{E}\big(Z(T)(X_1(T) - q)^+/B(T)\big) = \mathcal{U}^Y(T)\\
&\ge \Big(\mathbb{E}\big(Z(T)X_1(T)/B(T)\big) - q\,\mathbb{E}\big(Z(T)e^{-\int_0^T r(t)dt}\big)\Big)^+\\
&\ge \Big(\mathbb{E}\big(Z(T)X_1(T)/B(T)\big) - q\,e^{-rT}\,\mathbb{E}[\,Z(T)\,]\Big)^+\\
&\ge \Big(\mathbb{E}\big(Z(T)X_1(T)/B(T)\big) - q\,e^{-rT}\Big)^+ ,
\end{aligned}$$

thus

$$0 \le \mathcal{U}^Y(\infty) := \lim_{T\to\infty} \mathcal{U}^Y(T) = \lim_{T\to\infty} \downarrow \mathbb{E}\big(Z(T)X_1(T)/B(T)\big) < X_1(0). \tag{10.6}$$

The upper hedging price of the option is *strictly less* than the capitalization of the underlying stock at time $t=0$, and tends to $\mathcal{U}^Y(\infty) \in [0, X_1(0))$ as the time-horizon increases without limit.

If $\mathcal{M}$ is weakly diverse uniformly over some $[T_0,\infty)$, then the limit in (10.6) is actually zero: *The hedging price of a European call-option that can never be exercised, is equal to zero*. Indeed, for every fixed $p \in (0,1)$ and $T \ge \big(\frac{2\log n}{p\varepsilon\delta}\big) \vee T_0$, and with the normalization $X(0)=1$, the quantity

$$\mathbb{E}\bigg(\frac{Z(T)}{B(T)}X_1(T)\bigg) \le \mathbb{E}\bigg(\frac{Z(T)}{B(T)}V^{\mu}(T)\bigg) \le \mathbb{E}\bigg(\frac{Z(T)}{B(T)}V^{\mu^{(p)}}(T)\bigg)\, n^{\frac{1-p}{p}} e^{-\varepsilon\delta(1-p)T/2}$$

is dominated by $n^{\frac{1-p}{p}} e^{-\varepsilon\delta(1-p)T/2}$, from (7.2), (2.2) and the supermartingale property of the process $Z(\cdot)V^{\mu^{(p)}}(\cdot)/B(\cdot)$. Letting $T\to\infty$ we obtain $\mathcal{U}^Y(\infty)=0$. □

Remark 10.2. Note the sharp difference between this case and the situation where an equivalent martingale measure exists on every finite time horizon; namely, when both $Z(\cdot)$ and $Z(\cdot)X_1(\cdot)/B(\cdot)$ are martingales. Then we have $\mathbb{E}(Z(T)X_1(T)/B(T)) = X_1(0)$ for all $T\in(0,\infty)$, and $\mathcal{U}^Y(\infty) = X_1(0)$: as the time horizon increases without limit, the hedging price of the call option approaches the stock price at $t=0$ (see [KS] (1998), p. 62). □

Remark 10.3. The above theory extends to the case $d>n$ of *incomplete markets*, and more generally to *closed, convex constraints* on portfolio choice as in Chapter 5 of [KS] (1998), under the conditions of (6.4). The paper [KK] (2006) can be consulted for a treatment of these issues in a general semimartingale setting.

In particular, the Black-Scholes-type formula (10.5) can be generalized, in the spirit of (10.3), to the case $d>n$ and filtration $\mathbb{F}$ not necessarily equal to the Brownian filtration $\mathbb{F}^W$. Let Θ be the set of $\mathbb{F}-$progressively measurable processes $\theta(\cdot)$ that satisfy the requirements of (6.4); for each $\theta(\cdot)\in\Theta$, let us denote by $Z_\theta(\cdot)$ the process of (6.5). Then the upper hedging price of (10.2) is given as

$$\mathcal{U}^Y(T) = \sup_{\theta(\cdot)\in\Theta} \mathbb{E}\big(Y Z_\theta(T)/B(T)\big)\,. \tag{10.7}$$

□

Remark 10.4. Open Question: Develop a theory for pricing *American contingent claims* under the assumptions of the present section. As C. Kardaras (2006) observes, in the absence of an EMM it is *not* optimal to exercise an American call option (written on a non-dividend-paying stock) only at maturity $t=T$. Can one then characterize, or compute, the optimal exercise time? □

10.3 Utility Maximization in the Absence of EMM

Suppose we are given initial capital $w>0$, a time-horizon $[0,T]$ for some real $T>0$, and a utility function $u:(0,\infty)\to\mathbb{R}$ (strictly increasing, strictly concave, of class $\mathcal{C}^1$, with $u'(0) := \lim_{x\downarrow 0} u'(x) = \infty$, $u'(\infty) := \lim_{x\to\infty} u'(x) = 0$ and $u(0) := \lim_{x\downarrow 0} u(x)$). The problem is to compute the maximal expected utility from terminal wealth

$$\mathfrak{U}(w) := \sup_{h(\cdot)\in\mathcal{H}(w;T)} \mathbb{E}\big[\,u\big(\mathcal{V}^{w,h}(T)\big)\,\big]\,;$$

to decide whether the supremum is attained; and if so, to identify a strategy $\widehat{h}(\cdot)\in\mathcal{H}(w;T)$ that attains it. We place ourselves under the assumptions of the present section, including those of subsection 10.1 ($d=n$, $\mathbb{F}=\mathbb{F}^W$). □

Remark 10.5. The solution to this question is given by the replicating strategy $\widehat{h}(\cdot) \in \mathcal{H}_+(w;T)$ for the contingent claim

$$\Upsilon = I\big(\Xi(w)D(T)\big), \qquad \text{where} \quad D(t) := Z(t)/B(t) \quad \text{for} \;\; 0 \le t \le T\,,$$

in the sense $\mathcal{V}^{w,\widehat{h}}(T) = \Upsilon$ a.s. Here $Z(\cdot)$ is the exponential local martingale of (6.5), $I : (0,\infty) \to (0,\infty)$ is the inverse of the strictly decreasing marginal utility function $u' : (0,\infty) \to (0,\infty)$, and $\Xi : (0,\infty) \to (0,\infty)$ the inverse of the strictly decreasing function $\mathcal{W}(\cdot)$ given by

$$\mathcal{W}(\xi) := \mathbb{E}\big[\, D(T)\, I\,(\xi D(T))\,\big]\,, \quad 0 < \xi < \infty,$$

which we are assuming to be $(0,\infty)-$valued. $\square$

In the case of the *logarithmic utility function* $u(x) = \log x,\; x \in (0,\infty)$, it is easily shown that the "log-optimal" trading strategy $h^*(\cdot) \in \mathcal{H}_+(w;T)$ and its associated wealth process $V_*(\cdot) \equiv \mathcal{V}^{w,h^*}(\cdot)$ are given, respectively, by

$$h^*(t) = V_*(t)a^{-1}(t)\big[\, b(t) - r(t)\mathbf{I}\,\big], \quad V_*(t) = w/D(t) \tag{10.8}$$

for $0 \le t \le T$. The discounted log-optimal wealth process satisfies

$$d\big(V_*(t)/B(t)\big) = \big(V_*(t)/B(t)\big)\theta'(t)\big[\,\theta(t)\,dt + dW(t)\big]\,, \tag{10.9}$$

an equation whose solution is readily seen to be $V_*(t)/B(t) = w/Z(t)\,,\; 0 \le t \le T\,.$

Note that no assumption is been made regarding the existence of an equivalent martingale measure (EMM); to wit, $Z(\cdot)$ does not have to be a martingale. See Karatzas, Lehoczky, Shreve & Xu (1991) for more information on this problem and on its much more interesting *incomplete market* version $d > n$, under the assumption that the volatility matrix $\sigma(\cdot)$ is of full (row) rank and without assuming the existence of EMM.

Note also that the deflated optimal wealth process is constant: $\widehat{V}_*(\cdot) \equiv V_*(\cdot)Z(\cdot)/B(\cdot) = w\,.$ This should be contrasted to (6.12) of Remark 6.2, in the light of Remark 6.5.

The log-optimal trading strategy of (10.8) has some obviously desirable features, discussed in the next remark. But unlike the diversity-weighted portfolio of (7.1) or, more generally, the functionally generated portfolios of the next section, it needs for its implementation knowledge of the covariance structure and of the mean rates of return; these are quite hard to estimate in practice.

Remark 10.6. The "Numéraire" Property: Assume that the log-optimal strategy $h^*(\cdot) \in \mathcal{H}_+(w)$ of (10.8) is defined for all $0 \le t < \infty$; it has then the following *numéraire property*

$$\mathcal{V}^{w,h}(\cdot)/\mathcal{V}^{w,h^*}(\cdot) \quad \text{is a supermartingale,} \;\; \forall \;\; h(\cdot) \in \mathcal{H}_+(w), \tag{10.10}$$

and from this, one can derive the *asymptotic growth optimality* property

$$\limsup_{T\to\infty} \frac{1}{T} \log\left(\frac{\mathcal{V}^{w,h}(T)}{\mathcal{V}^{w,h^*}(T)}\right) \le 0 \quad \text{a.s.,} \quad \forall \;\; h(\cdot) \in \mathcal{H}_+(w)\,.$$

These are the same notions we encountered in Problem 6 of Section 4, in the setup of portfolios (as opposed to trading strategies). For a detailed study of these issues in a far more general context, see [KK] (2007). $\square$

Remark 10.7. (Platen 2006): The equation for $\Psi(\cdot) := V_*(\cdot)/B(\cdot) = w/Z(\cdot)$ in (10.9) is

$$d\Psi(t) = \alpha(t)\,dt + \sqrt{\Psi(t)\alpha(t)}\,d\mathfrak{B}(t), \qquad \Psi(0) = w$$

with $\mathfrak{B}(\cdot)$ a one-dimensional Brownian motion and $\alpha(t) := \Psi(\cdot)\|\theta(\cdot)\|^2$.

Then $\Psi(\cdot)$ is a time-changed and scaled *squared Bessel process in dimension 4* (sum of squares of four independent Brownian motions); that is, $\Psi(\cdot) = \mathfrak{X}\big(A(\cdot)\big)/4$, where

$$A(\cdot) := \int_0^{\cdot} \alpha(s)\,ds \qquad \text{and} \qquad \mathfrak{X}(u) = 4(w+u) + 2\int_0^u \sqrt{\mathfrak{X}(v)}\,d\mathfrak{b}(v), \quad u \ge 0$$

in terms of yet another standard, one-dimensional Brownian motion $\mathfrak{b}(\cdot)$. $\square$

Remark 10.8. It might be useful to note at this point that, just as for the optimization problems of this subsection, no assumption regarding the existence of EMM was necessary for any of the Problems 1-6 of Section 4. □

Chapter III
Functionally Generated Portfolios

Functionally generated portfolios were introduced in Fernholz (1999.a), and generalize broadly the diversity-weighted portfolios of Section 7. For this new class of portfolios one can derive a decomposition of their relative return analogous to that of (7.5), and this proves useful in the construction and study of arbitrages relative to the market. Just like (7.5), this new decomposition (11.2) does not involve stochastic integrals, and opens the possibility for making probability-one comparisons over given, fixed time-horizons. Functionally generated portfolios can be constructed for general classes of assets, with the market portfolio replaced by an arbitrary passive portfolio of the assets under consideration.

11 Portfolio generating functions

Certain real-valued functions of the market weights $\mu_1(t), \ldots, \mu_n(t)$ can be used to construct dynamic portfolios that behave in a controlled manner. The *portfolio generating functions* that interest us most fall into two categories: smooth functions of the market weights, and smooth functions of the ranked market weights. Those portfolio generating functions that are smooth functions of the market weights can be used to create portfolios with returns that satisfy almost sure relationships relative to the market portfolio, and, hence, can be applied to situations in which arbitrage might be possible. Those functions that are smooth functions of the ranked market weights can be used to analyze the role of company *size* in portfolio behavior.

Suppose we are given a function $\mathbf{G} : U \to (0, \infty)$ which is defined and of class $\mathcal{C}^2$ on some open neighborhood U of Δ^n_+, and such that the mapping $x \mapsto x_i D_i \log \mathbf{G}(x)$ is bounded on U for all $i = 1, \cdots, n$. Consider also the portfolio $\pi(\cdot)$ with weights

$$\pi_i(t) = \Big(D_i \log \mathbf{G}(\mu(t)) + 1 - \sum_{j=1}^{n} \mu_j(t) D_j \log \mathbf{G}(\mu(t)) \Big) \cdot \mu_i(t) \,, \qquad 1 \le i \le n \,. \tag{11.1}$$

We call this the *portfolio generated by* $\mathbf{G}(\cdot)$. It can be shown that the relative wealth process of this portfolio, with respect to the market, is given by the *master formula*

$$\log \left(\frac{V^\pi(T)}{V^\mu(T)} \right) \; = \; \log \left(\frac{\mathbf{G}(\mu(T))}{\mathbf{G}(\mu(0))} \right) + \int_0^T \mathfrak{g}(t) \, dt \,, \qquad 0 \le T < \infty \,, \tag{11.2}$$

where the so-called *drift process* $\mathfrak{g}(\cdot)$ is given by

$$\mathfrak{g}(t) \; := \; \frac{-1}{2\mathbf{G}(\mu(t))} \sum_{i=1}^{n} \sum_{j=1}^{n} D^2_{ij} \mathbf{G}(\mu(t)) \, \mu_i(t) \mu_j(t) \tau^\mu_{ij}(t) \,. \tag{11.3}$$

The portfolio weights of (11.1) depend only on the market weights $\mu_1(t), \cdots, \mu_n(t)$, not on the covariance structure of the market. Thus the portfolio of (11.1) can be implemented, and its associated wealth process $V^\pi(\cdot)$ observed through time, only in terms of the evolution of these market weights over $[0, T]$.

The covariance structure enters only in the computation of the drift term in (11.3). But the remarkable thing is that, in order to compute the cumulative effect $\int_0^T \mathfrak{g}(t)\,dt$ of this drift, there is no need to know or estimate this covariance structure at all; (11.2) does this for us, in the form $\int_0^T \mathfrak{g}(t)\,dt = \log\big(V^\pi(T)\mathbf{G}(\mu(0))/V^\mu(T)\mathbf{G}(\mu(T))\big)$, and in terms of quantities that are observable.

The proof of the very important "master formula" (11.2) is given below, at the very end of the present section. It can be skipped on first reading.

Remark 11.1. Suppose the function $\mathbf{G}(\cdot)$ is *concave*, or, more precisely, its Hessian $D^2\mathbf{G}(x) = \big(D^2_{ij}\mathbf{G}(x)\big)_{1\le i,j\le n}$ has at most one positive eigenvalue for each $x \in U$ and, if a positive eigenvalue exists, the corresponding eigenvector is orthogonal to Δ^n_+ . Then the portfolio $\pi(\cdot)$ generated by $\mathbf{G}(\cdot)$ as in (11.1) is long-only (i.e., each weight $\pi_i(\cdot)$ is non-negative), and the drift term $\mathfrak{g}(\cdot)$ is non-negative; if $\mathrm{rank}\big(D^2\mathbf{G}(x)\big) > 1$ holds for each $x \in U$, then $\mathfrak{g}(\cdot)$ is positive.

For instance:

1. $\mathbf{G}(\cdot) \equiv w$, a positive constant, generates the *market* portfolio;

2. $\mathbf{G}(x) = w_1x_1+\cdots+w_nx_n$ generates the *passive* portfolio that buys at time $t = 0$, and holds up until time $t = T$, a fixed number of shares w_i in each stock $i = 1, \cdots , n$ (the market portfolio corresponds to the special case $w_1 = \cdots = w_n = w$ of equal numbers of shares across assets);

3. $\mathbf{G}(x) \equiv \mathbf{G}_p(x) := \big(x_1^p + \cdots + x_n^p\big)^{1/p}$, for some $0 < p < 1$, generates the *diversity-weighted* portfolio $\mu^{(p)}(\cdot)$ of (7.1), with drift-process $\mathfrak{g}(\cdot) \equiv (1-p)\gamma^*_{\mu^{(p)}}(\cdot)$; and

4. $\mathbf{G}(x) \equiv \mathbf{F}(x) := \big(x_1\cdots x_n\big)^{1/n}$, generates the *equally weighted* portfolio $\varphi_i(\cdot) \equiv 1/n$, $i = 1, \cdots , n$ introduced in Remark 7.2, with drift $\mathfrak{g}^\varphi(\cdot) \equiv \gamma^*_\varphi(\cdot)$ as in (7.11).

   In a similar manner, $\mathbf{F}_c(x) := c + \mathbf{F}(x)$, for $c \in (0,\infty)$, generates the convex combination

$$\varphi^c_i(t) := \frac{\mathbf{F}(\mu(t))}{c+\mathbf{F}(\mu(t))}\cdot\frac{1}{n} + \frac{c}{c+\mathbf{F}(\mu(t))}\cdot\mu_i(t)\,, \qquad i = 1,\cdots,n \tag{11.4}$$

   of the equally weighted portfolio and the market, with associated drift-rate

$$\mathfrak{g}^{\varphi^c}(t) = \frac{\mathbf{F}(\mu(t))}{c+\mathbf{F}(\mu(t))}\,\gamma^*_\varphi(t)\,. \tag{11.5}$$

5. Consider now the *entropy function* $\mathbf{H}(x) := -\sum_{i=1}^n x_i \log x_i,\ \ x \in \Delta^n_+$ and, for any given $c \in (0,\infty)$, its modification

$$\mathbf{H}_c(x) := c + \mathbf{H}(x), \quad \text{which satisfies:} \quad c < \mathbf{H}_c(x) \le c + \log n, \ \ x \in \Delta^n_+\,. \tag{11.6}$$

   This modified entropy function generates an *entropy-weighted* portfolio $\pi^c(\cdot)$ with weights and drift-process given, respectively, as

$$\pi^c_i(t) = \frac{\mu_i(t)}{\mathbf{H}_c(\mu(t))}\big(c - \log\mu_i(t)\big), \ \ 1 \le i \le n \quad \text{and} \quad \mathfrak{g}^c(t) = \frac{\gamma^*_\mu(t)}{\mathbf{H}_c(\mu(t))}\,. \tag{11.7}$$

□

To obtain some idea about the behavior of one of these portfolios with actual stocks, we ran a simulation of a diversity-weighted portfolio using the stock database from the Center for Research in Securities Prices (CRSP) at the University of Chicago. The data included 50 years of monthly values from 1956 to 2005 for exchange-traded stocks after the removal of closed-end funds, REITs, and ADRs not included in the S&P 500 Index. From this universe, we considered a cap-weighted

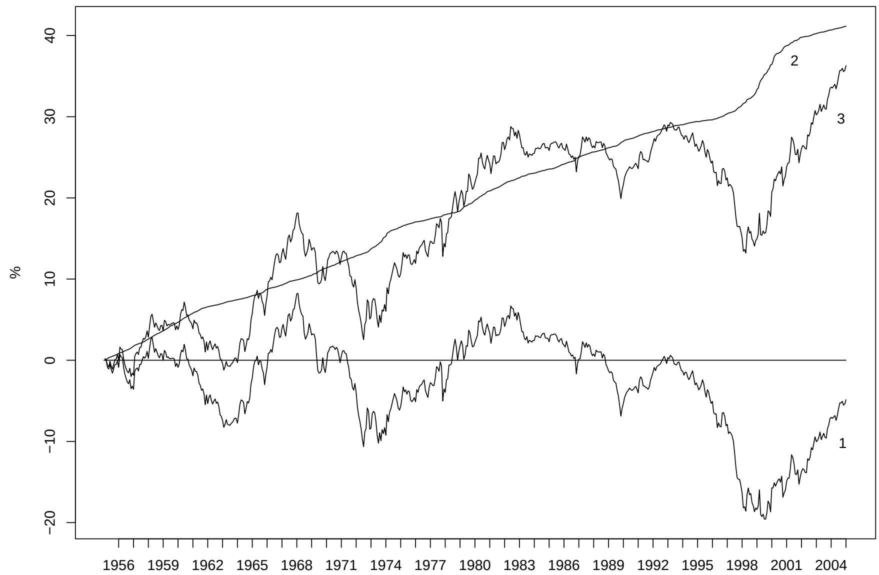

Figure 11.2: Simulation of a $\mathbf{G}_p$-weighted portfolio, 1956–2005
1: generating function; 2: drift process; 3: relative return.

large-stock index consisting of the largest 1000 stocks in the database. Against this index, we simulated the performance of the corresponding diversity-weighted portfolio, generated by $\mathbf{G}_p$ of Remark 11.1, Example 3 above, with $p = 1/2$. No trading costs were included.

The results of the simulation are presented in Figure 11.2: Curve 1 is the change in the generating function, Curve 2 is the cumulative drift process $\int_0^{\cdot} \mathfrak{g}(t)\, dt$, and Curve 3 is the relative return. Each curve shows the cumulative value of the monthly changes induced in the corresponding process by capital gains or losses in the stocks, so the curves are unaffected by monthly changes in the composition of the database. As can be seen, Curve 3 is the sum of Curves 1 and 2. The cumulative drift process $\int_0^{\cdot} \mathfrak{g}(t)\, dt$ was the dominant term over the period, with a total contribution of about 40 percentage points to the relative return. The drift process $\mathfrak{g}(\cdot)$ was quite stable over the 50-year period, with the possible exception of the period around 2000, when "irrational exuberance" increased the volatility of the stocks as well as the intrinsic volatility of the entire market and, hence, increased the value of $\mathfrak{g}(\cdot) \equiv (1-p)\gamma^*_{\mu^{(p)}}(\cdot)$. The cumulative drift process $\int_0^{\cdot} \mathfrak{g}(t)\, dt$ here has been adjusted to account for "leakage"; see Remark 11.9 below.

11.1 Sufficient Intrinsic Volatility leads to Arbitrage

Broadly accepted practitioner wisdom upholds that *sufficient volatility creates growth opportunities in a financial market.*

We have already encountered an instance of this phenomenon in Remark 3.2; we saw there that, in the presence of a strong non-degeneracy condition on the market's covariance structure, "reasonably diversified" long-only portfolios with constant weights represent superior long-term growth opportunities relative to the overall market.

We shall examine in Example 11.1 below another instance of this phenomenon. We shall try again to put the above intuition on a precise quantitative basis, by identifying now the *excess growth rate of the market portfolio* – which also measures the market's *intrinsic volatility*, according to (3.8) and the discussion following it – as a *driver of growth*; to wit, as a quantity whose 'availability' or 'sufficiency' (boundedness away from zero) can lead to opportunities for strong arbitrage and for superior long-term growth, relative to the market.

Example 11.1. Suppose now that in the market $\mathcal{M}$ there exist real constants $\zeta > 0$, $T > 0$ such that
$$\frac{1}{T}\int_0^T \gamma_\mu^*(t)\,dt \ge \zeta \tag{11.8}$$
holds almost surely. For instance, this is the case when the excess growth rate of the market portfolio is bounded away from zero: that is, when we have almost surely
$$\gamma_\mu^*(t) \ge \zeta, \quad \forall\ \ 0 \le t \le T\,. \tag{11.9}$$
Consider again the entropy-weighted portfolio $\pi^c(\cdot)$ of (11.7), namely
$$\pi_i^c(t) \;=\; \frac{\mu_i(t)\big(c-\log\mu_i(t)\big)}{\sum_{j=1}^n \mu_j(t)\big(c-\log\mu_j(t)\big)}\,, \qquad i=1,\cdots,n\,, \tag{11.10}$$
now written in a form that makes plain its over-weighting of the small capitalization stocks, relative to the market portfolio. From (11.2), (11.7) and the inequalities of (11.6), one sees that the portfolio $\pi^c(\cdot)$ in (11.7) satisfies
$$\begin{aligned}\log\left(\frac{V^{\pi^c}(T)}{V^\mu(T)}\right) \;&=\; \log\left(\frac{\mathbf{H}_c(\mu(T))}{\mathbf{H}_c(\mu(0))}\right)+\int_0^T \frac{\gamma_\mu^*(t)}{\mathbf{H}_c(\mu(t))}\,dt \\ &>\; -\log\left(\frac{c+\mathbf{H}\big(\mu(0)\big)}{c}\right)+\frac{\zeta T}{c+\log n}\end{aligned} \tag{11.11}$$
almost surely. Thus, for every time horizon $[0,T]$ of length
$$T > \mathcal{T}_*(c) := \frac{1}{\zeta}\,\big(c+\log n\big)\log\left(\frac{c+\mathbf{H}\big(\mu(0)\big)}{c}\right)\,,$$
or for that matter every
$$T > \mathcal{T}_* = \frac{1}{\zeta}\,\mathbf{H}\big(\mu(0)\big) \tag{11.12}$$
(since $\lim_{c\to\infty}\mathcal{T}_*(c)=\mathcal{T}_*$), and for $c>0$ sufficiently large, the portfolio $\pi^c(\cdot)$ of (11.7) satisfies the condition $\mathbb{P}(V^{\pi^c} > V^\mu(T))=1$ for strong arbitrage relative to the market $\mu(\cdot)$, on the given time horizon $[0,T]$. It is straightforward that (6.3) is also satisfied, with $q=c/(c+\mathbf{H}(\mu(0))$. $\square$

In particular, with the notation of (6.2) we have almost surely $\mathcal{L}^{\pi^c,\mu} \ge \zeta/(c+\log n) > 0$ (the condition for superior long-term growth for $\pi^c(\cdot)$ relative to the market $\mu(\cdot)$), provided that (11.9) holds for *all* sufficiently long time-horizons $T>0$.

It should also be noted that *we have not imposed in the discussion of Example 11.1* **any** *assumption on the volatility structure of the market (such as (1.15), (1.16) or (3.10)) beyond the absolutely minimal condition of (1.2).*

Figure 11.3 shows the cumulative excess growth $\int_0^\cdot \gamma_\mu^*(t)\,dt$ for the U.S. equities market over most of the twentieth century. Note the conspicuous bumps in the curve, first in the Great Depression period in the early 1930s, then again in the "irrational exuberance" period at the end of the century. The data used for this chart come from the monthly stock database of the Center for Research in Securities Prices (CRSP) at the University of Chicago. The market we construct consists of the stocks traded on the New York Stock Exchange (NYSE), the American Stock Exchange (AMEX) and the NASDAQ Stock Market, after the removal of all REITs, all closed-end funds, and those ADRs not included in the S&P 500 Index. Until 1962, the CRSP data included only NYSE stocks. The AMEX stocks were included after July 1962, and the NASDAQ stocks were included at the beginning of 1973. The number of stocks in this market varies from a few hundred in 1927 to about 7500 in 2005.

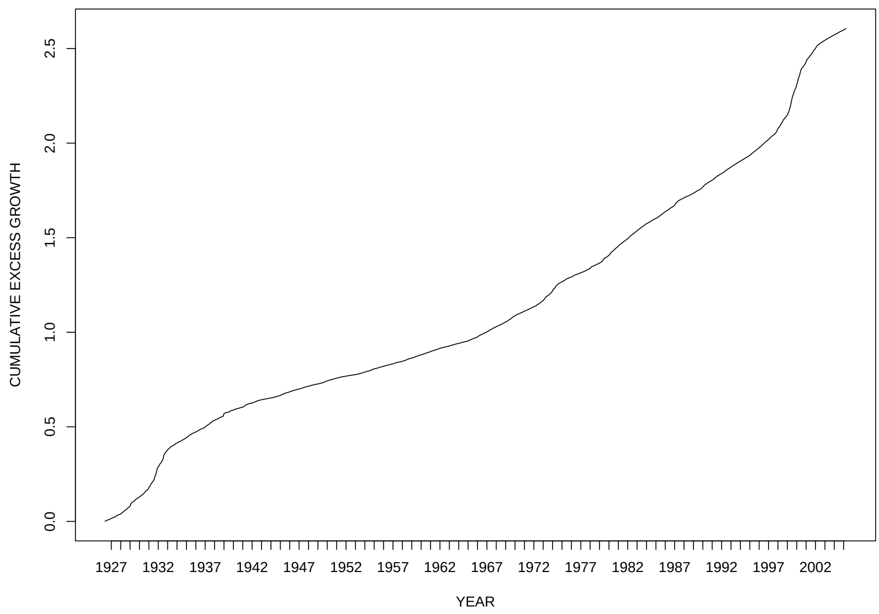

Figure 11.3: Cumulative excess growth $\int_0^{\cdot} \gamma^*_\mu(t)dt$.
U.S. market, 1927–2005

This computation for Figure 11.3 does not need any estimation of covariance structure: from (11.11) we can express this cumulative excess growth

$$\int_0^{\cdot} \gamma^*_\mu(t)\,dt = \int_0^{\cdot} \mathbf{H}_c\big(\mu(t)\big)\,d\log\left(\frac{V^{\pi^c}(t)\,\mathbf{H}_c(\mu(0))}{V^\mu(t)\,\mathbf{H}_c(\mu(t))}\right)\,,$$

just in terms of quantities that are observable in the market. The plot suggests that the U.S. market has exhibited a strictly increasing cumulative excess growth over this period.

Remark 11.2. Let us recall here our discussion of the conditions in (5.3): if the covariance matrix $a(\cdot)$ has all its eigenvalues bounded away from both zero and infinity, then the condition (11.9) (respectively, (11.8)) is equivalent to diversity (respectively, weak diversity) on $[0,T]$. The point of these conditions is that they guarantee the existence of strong arbitrage relative to the market, even when volatilities are unbounded and diversity fails. In the next section we shall study a concrete example of such a situation. □

Remark 11.3. Open Question: From (11.11) it is not difficult to see that if we are allowed to start with the market arbitrarily close to the "boundary", i.e., if $\mu(0)$ can be chosen such that $\mathbf{H}(\mu(0))$ is arbitrarily small, then condition (11.9) will assure the existence of short-term arbitrage (as opposed to arbitrage over sufficiently long time intervals). Suppose now that the market can reach a point arbitrarily close to the boundary in an arbitrarily short time with positive probability. We could then use the strategy of holding the market portfolio until we arrive close enough to the boundary—which will occur, at least with positive probability—and then switch to the arbitrage portfolio, so short-term arbitrage will again be possible. However, *strong arbitrage*, in the sense that

$$\mathbb{P}\,[\,V^\pi(T) > V^\mu(T)\,] = 1$$

in (6.1), cannot be assured by this argument. Indeed, it seems to be an open problem whether or not condition (11.9) implies strong arbitrage relative to the market over arbitrarily short time periods. □

Remark 11.4. Example and Open Questions: For $0 < p \leq 1$, the quantity

$$\gamma^*_{\pi,p}(t) := \frac{1}{2} \sum_{i=1}^{n} \big(\pi_i(t)\big)^p \, \tau^\pi_{ii}(t) \tag{11.13}$$

generalizes the excess growth rate of a portfolio $\pi(\cdot)$, in the sense that $\gamma^*_{\pi,1}(\cdot) \equiv \gamma^*_\pi(\cdot)$. With $0 < p < 1$, consider the a.s. requirement

$$\Gamma(T) \leq \int_0^T \gamma^*_{p,\mu}(t)\, dt < \infty, \qquad \forall \;\; 0 \leq T < \infty, \tag{11.14}$$

for some continuous, strictly increasing function $\Gamma : [0,\infty) \to [0,\infty)$ with $\Gamma(0) = 0$, $\Gamma(\infty) = \infty$. As shown in Proposition 3.8 of [FK] (2005), the condition (11.14) guarantees that the portfolio

$$\pi_i(t) := p \cdot \frac{\big(\mu_i(t)\big)^p}{\sum_{j=1}^n \big(\mu_j(t)\big)^p} + (1-p) \cdot \mu_i(t), \qquad i = 1, \cdots, n \tag{11.15}$$

is a strong arbitrage opportunity relative to the market, namely, that $\mathbb{P}\,[\,V^\pi(T) > V^\mu(T)\,] = 1$ holds over sufficiently long time-horizons: $T > \Gamma^{-1}\big((1/p) n^{1-p} \log n\big)$.

Note that the portfolio of (11.15) is a convex combination, with fixed weights $1-p$ and p, of the market and of its diversity-weighted index $\mu^{(p)}(\cdot)$ in (7.1), respectively.

Some questions suggest themselves:

- Does (11.14) guarantee the existence of relative arbitrage opportunities over arbitrary time-horizons?
- Is there a result on the existence of relative arbitrage, that generalizes both Example 11.1 and the result outlined in (11.14), (11.15)?
- What quantity, or quantities, might then be involved, in place of the market excess growth or its generalization (11.14)? Is there a “best” result of this type? □

Example 11.2. Equal Weighting: Recall the computation (7.11) for the excess growth rate of the equally weighted portfolio $\varphi_i(\cdot) \equiv 1/n$, $i = 1, \cdots n$, and suppose that

$$\big(\mu_1(t) \cdots \mu_n(t)\big)^{1/n} \, \gamma^*_\varphi(t) \geq \zeta, \qquad 0 \leq t \leq T \tag{11.16}$$

holds a.s., for some real constant $\zeta > 0$.

Recall also the modification $\varphi^c(\cdot)$ of this portfolio, as in (11.4); this is generated by the function $\mathbf{F}_c(x) = c + \mathbf{F}(x)$, with $c > 0$ and $\mathbf{F}(x) := (x_1 \cdots x_n)^{1/n} \in (0, n^{-1/n}]$, $x \in \Delta^n_+$. From (11.5) and (11.2), we deduce the a.s. comparisons

$$\log\left(\frac{V^{\varphi^c}(T)}{V^\mu(T)}\right) = \log\left(\frac{c + \mathbf{F}\big(\mu(T)\big)}{c + \mathbf{F}\big(\mu(0)\big)}\right) + \int_0^T \frac{\mathbf{F}\big(\mu(t)\big)\, \gamma^*_\varphi(t)}{c + \mathbf{F}\big(\mu(t)\big)}\, dt$$

and

$$\log\left(\frac{V^{\varphi^c}(T)}{V^\mu(T)}\right) \geq \log\left(\frac{c}{c + n^{-1/n}}\right) + \frac{\zeta\, T}{c + n^{-1/n}} \tag{11.17}$$

for the portfolio $\varphi^c(\cdot)$ of (11.4). Therefore, we have $\mathbb{P}\big(V^{\varphi^c}(T) > V^\mu(T)\big) = 1$, provided that $T > \frac{1}{\zeta}\,\big(c + n^{-1/n}\big) \cdot \log\big(c/(c + n^{-1/n})\big)$. Consequently, if the time horizon is sufficiently long, to wit

$$T > T_* := \frac{1}{\zeta}\, n^{-1/n},$$

there exists a number $c \in (0,\infty)$ such that the market-modulated equally weighted portfolio $\varphi^c(\cdot)$ of (11.4) is a strong arbitrage relative to the market. □

Remark 11.5. Open Question: We have presented a few portfolios that lead to arbitrage relative to the market; they are all functionally generated. Is there a "best" such example within that class? Are there similar examples of portfolios that are *not* functionally generated, nor trivial modifications thereof? How representative (or "dense") in this context is the class of functionally generated portfolios? □

Remark 11.6. Open Question: Generalize the theory of functionally generated portfolios to the case of a market with a countable infinity ($n = \infty$) of assets, or to some other model with a variable, unbounded number of assets. □

Remark 11.7. Open Question: What, if any, is the connection of functionally generated portfolios with the "universal portfolios" of Cover (1991) and Jamshidian (1992)? □

11.2 Rank, Leakage, and the Size Effect

An important generalization of the ideas and methods in this section concerns generating functions that record market weights not according to their name (or index) i, but according to their rank. To present this generalization, let us start by recalling the order statistics notation of (1.18), and consider for each $0 \le t < \infty$ the random permutation $\big(p_t(1), \cdots, p_t(n)\big)$ of $(1, \cdots, n)$ with

$$\mu_{p_t(k)}(t) = \mu_{(k)}(t), \qquad \text{and} \quad p_t(k) < p_t(k+1) \quad \text{if} \quad \mu_{(k)}(t) = \mu_{(k+1)}(t) \tag{11.18}$$

for $k = 1, \ldots, n$. In words: $p_t(k)$ is the name (index) of the stock that occupies the k^{th} rank in terms of relative capitalization at time t; ties are resolved by resorting to the lowest index.

Using Itô's rule for convex functions of semimartingales (e.g. [KS] (1991), section 3.7), one can obtain the following analogue of (2.5) for the *ranked market-weights*

$$\begin{aligned} \frac{d\mu_{(k)}(t)}{\mu_{(k)}(t)} = &\Big(\gamma_{p_t(k)}(t) - \gamma^{\mu}(t) + \frac{1}{2}\tau^{\mu}_{(kk)}(t)\Big)\, dt + \frac{1}{2}\Big(d\mathfrak{L}^{k,k+1}(t) - d\mathfrak{L}^{k-1,k}(t)\Big) \\ &+ \sum_{\nu=1}^{d} \big(\sigma_{p_t(k)\nu}(t) - \sigma^{\mu}_{\nu}(t)\big)\, dW_{\nu}(t) \end{aligned} \tag{11.19}$$

for each $k = 1, \ldots, n-1$. Here the quantity $\mathfrak{L}^{k,k+1}(t) \equiv \Lambda_{\Xi_k}(t)$ is the semimartingale local time at the origin, accumulated by the non-negative process

$$\Xi_k(t) := \log\big(\mu_{(k)}/\mu_{(k+1)}\big)(t), \qquad 0 \le t < \infty \tag{11.20}$$

up to the calendar time t; it measures the cumulative effect of the changes that have occurred during the time-interval $[0, t]$ between ranks k and $k+1$. We are also setting $\mathfrak{L}^{0,1}(\cdot) \equiv 0$, $\mathfrak{L}^{m,m+1}(\cdot) \equiv 0$ and $\tau^{\mu}_{(k\ell)}(\cdot) := \tau^{\mu}_{p_t(k)p_t(\ell)}(\cdot)$.

A derivation of this result, under appropriate conditions that we choose not to broach here, can be found on pp. 76-79 of Fernholz (2002); see also Banner & Ghomrasni (2007) for generalizations.

With this setup, we have then the following generalization of the "master equation" (11.2): Consider a function $\mathbf{G} : U \to (0, \infty)$ exactly as assumed there, written in the form

$$\mathbf{G}(x_1, \cdots, x_n) = \mathfrak{G}\big(x_{(1)}, \cdots, x_{(n)}\big), \qquad \forall \quad x \in U$$

for some $\mathfrak{G} \in \mathcal{C}^2(U)$ and U an open neighborhood of Δ^n_+. Introduce the shorthand

$$x_{(\cdot)} := \big(x_{(1)}, \cdots, x_{(n)}\big)', \qquad \mu_{(\cdot)}(t) := \big(\mu_{(1)}(t), \cdots, \mu_{(n)}(t)\big)', \qquad \tau^{\mu}_{(k\ell)}(t) := \tau^{\mu}_{p_t(k)p_t(\ell)}(t)$$

as well as the notation

$$\Gamma(T) := -\int_0^T \frac{1}{2\mathfrak{G}\big(\mu_{(\cdot)}(t)\big)} \sum_{k=1}^n \sum_{\ell=1}^n D^2_{k\ell}\, \mathfrak{G}\big(\mu_{(\cdot)}(t)\big)\, \mu_{(k)}(t)\mu_{(\ell)}(t)\, \tau^{\mu}_{(k\ell)}(t)\, dt \\ + \frac{1}{2}\sum_{k=1}^{n-1} \big(\underline{\pi}_{p_t(k+1)}(t) - \underline{\pi}_{p_t(k)}(t)\big)\, d\mathfrak{L}^{k,k+1}(t)\,. \tag{11.21}$$

Then it can be shown that the performance of the portfolio $\underline{\pi}(\cdot)$ given as

$$\underline{\pi}_{p_t(k)}(t) = \left(D_k \log \mathfrak{G}\big(\mu_{(\cdot)}(t)\big) + 1 - \sum_{\ell=1}^n \mu_{(\ell)}(t) D_\ell \log \mathfrak{G}\big(\mu_{(\cdot)}(t)\big) \right) \mu_{(k)}(t) \tag{11.22}$$

for $1 \le k \le n$, relative to the market, is

$$\log\left(\frac{V^{\underline{\pi}}(T)}{V^{\mu}(T)}\right) = \log\left(\frac{\mathfrak{G}\big(\mu_{(\cdot)}(T)\big)}{\mathfrak{G}\big(\mu_{(\cdot)}(0)\big)}\right) + \Gamma(T)\,, \qquad 0 \le T < \infty\,. \tag{11.23}$$

We say that $\underline{\pi}(\cdot)$ is the portfolio generated by the function $\mathfrak{G}(\cdot)$. The detailed proof can be found in Fernholz (2002), pp. 79-83.

For instance, $\mathfrak{G}(x_{(\cdot)}) = x_{(1)}$ generates the portfolio $\underline{\pi}_{p_t(k)}(t) = \delta_{1k}\,,\ k = 1, \cdots, n\,,\ 0 \le t < \infty$ that invests only in the largest stock, at all times. The relative performance

$$\log\left(\frac{V^{\underline{\pi}}(T)}{V^{\mu}(T)}\right) = \log\left(\frac{\mu_{(1)}(T)}{\mu_{(1)}(0)}\right) - \frac{1}{2}\,\mathfrak{L}^{1,2}(T), \quad 0 \le T < \infty$$

of this portfolio will suffer in the long run, if there are many changes in leadership: in order for the biggest stock to do well relative to the market, it must crush all competition!

Example 11.3. The Size Effect: This is the tendency of small stocks to have higher long-term returns relative to their larger brethren. The formula of (11.23) offers a simple, structural explanation of this observed phenomenon, as follows.

Fix an integer $m \in \{2, \cdots, n-1\}$ and consider the functions $\mathcal{G}_L(x) = x_{(1)} + \cdots + x_{(m)}$ and $\mathcal{G}_S(x) = x_{(m+1)} + \cdots + x_{(n)}$. These generate, respectively, a *large-stock portfolio*

$$\zeta_{p_t(k)}(t) = \frac{\mu_{(k)}(t)}{\mathcal{G}_L(\mu(t))}\,, \quad k = 1, \cdots, m \quad \text{and} \quad \zeta_{p_t(k)}(t) = 0, \quad k = m+1, \cdots, n \tag{11.24}$$

and a *small-stock portfolio*

$$\eta_{p_t(k)}(t) = \frac{\mu_{(k)}(t)}{\mathcal{G}_S(\mu(t))}\,, \quad k = m+1, \cdots, n \quad \text{and} \quad \eta_{p_t(k)}(t) = 0, \quad k = 1, \cdots, m. \tag{11.25}$$

According to (11.23) and (11.22), the performances of these portfolios, relative to the market, are given by

$$\log\left(\frac{V^{\zeta}(T)}{V^{\mu}(T)}\right) = \log\left(\frac{\mathcal{G}_L(\mu(T))}{\mathcal{G}_L(\mu(0))}\right) - \frac{1}{2}\int_0^T \zeta_{(m)}(t)\, d\mathfrak{L}^{m,m+1}(t), \tag{11.26}$$

$$\log\left(\frac{V^{\eta}(T)}{V^{\mu}(T)}\right) = \log\left(\frac{\mathcal{G}_S(\mu(T))}{\mathcal{G}_S(\mu(0))}\right) + \frac{1}{2}\int_0^T \eta_{(m)}(t)\, d\mathfrak{L}^{m,m+1}(t), \tag{11.27}$$

respectively. Therefore,

$$\log\left(\frac{V^{\eta}(T)}{V^{\zeta}(T)}\right) = \log\left(\frac{\mathcal{G}_S(\mu(T))\,\mathcal{G}_L(\mu(0))}{\mathcal{G}_L(\mu(T))\,\mathcal{G}_S(\mu(0))}\right) + \int_0^T \frac{\zeta_{(m)}(t) + \eta_{(m)}(t)}{2}\, d\mathfrak{L}^{m,m+1}(t). \tag{11.28}$$

If there is “stability” in the market, in the sense that the ratio of the relative capitalization of small to large stocks remains stable over time, then the first term on the right-hand side of (11.28) does not change much, whereas the second term keeps increasing and accounts for the better relative performance of the small stocks. Note that this argument does not need to invoke any assumption about the putative greater riskiness of the smaller stocks at all.

The paper Fernholz & Karatzas (2006) studies conditions under which such stability in relative capitalizations prevails, and contains further discussion related to the “liquidity premium” for equities. □

Remark 11.8. Estimation of Local Times: Hard as this might be to have guessed from the outset, the local times $\mathfrak{L}^{k,k+1}(\cdot) \equiv \Lambda_{\Xi_k}(\cdot)$ appearing in (11.19), (11.21) can be estimated in practice quite accurately; indeed, (11.26) gives

$$\mathfrak{L}^{m,m+1}(\cdot) = \int_0^{\cdot} \frac{2}{\zeta_{(m)}(t)} \, d\log\left(\frac{\mathcal{G}_L(\mu(t))}{\mathcal{G}_L(\mu(0))}\frac{V^{\mu}(t)}{V^{\zeta}(t)}\right), \quad m = 1, \cdots, n-1, \tag{11.29}$$

and the quantity on the right-hand side is completely observable. □

Remark 11.9. Leakage in a Diversity-Weighted Index of Large Stocks: With the integer m and the large-stock portfolio $\zeta(\cdot)$ as in Example 11.3, and a fixed number $r \in (0,1)$, consider the *diversity-weighted, large-stock portfolio*

$$\mu^{\sharp}_{p_t(k)}(t) = \frac{\big(\mu_{(k)}(t)\big)^r}{\sum_{\ell=1}^m \big(\mu_{(\ell)}(t)\big)^r}\,, \quad 1 \le k \le m \qquad \text{and} \qquad \mu^{\sharp}_{p_t(k)}(t) = 0, \quad m+1 \le k \le n \tag{11.30}$$

generated by the function $\mathbb{G}_r(x) = \Big(\sum_{\ell=1}^m \big(x_{(\ell)}\big)^r\Big)^{1/r}$, by analogy with (7.4), (7.1). Then

$$\log\left(\frac{V^{\mu^\sharp}(T)}{V^{\mu}(T)}\right) = \log\left(\frac{\mathbb{G}_r(\mu(T))}{\mathbb{G}_r(\mu(0))}\right) + (1-r)\int_0^T \gamma^*_{\mu^\sharp}(t)\,dt - \int_0^T \frac{\mu^\sharp_{(m)}(t)}{2}\, d\mathfrak{L}^{m,m+1}(t)$$

gives the performance of the portfolio in (11.30) relative to the market, and

$$\begin{aligned}\log\left(\frac{V^{\mu^\sharp}(T)}{V^{\zeta}(T)}\right) &= \log\left(\frac{\mathbf{G}_r\big(\zeta_{(1)}(T),\cdots,\zeta_{(m)}(T)\big)}{\mathbf{G}_r\big(\zeta_{(1)}(0),\cdots,\zeta_{(m)}(0)\big)}\right) + (1-r)\int_0^T \gamma^*_{\mu^\sharp}(t)\,dt \\ &\quad - \frac{1}{2}\int_0^T \big(\mu^\sharp_{(m)}(t) - \zeta_{(m)}(t)\big)\, d\mathfrak{L}^{m,m+1}(t)\end{aligned} \tag{11.31}$$

gives the performance of (11.30) relative to the large-stock portfolio $\zeta(\cdot)$ of (11.24). We have used here the scale-invariance property

$$\frac{\mathbf{G}_r(x_1,\cdots,x_n)}{x_1+\cdots+x_n} = \mathbf{G}_r\left(\frac{x_1}{x_1+\cdots+x_n},\cdots,\frac{x_n}{x_1+\cdots+x_n}\right)$$

of the diversity function $\mathbf{G}_r(\cdot)$ in (7.4) for $0 < r < 1$, which implies the reduction

$$\frac{\mathbb{G}_r(\mu(t))}{\mathcal{G}_L(\mu(t))} = \mathbf{G}_r\big(\zeta_{(1)}(t),\cdots,\zeta_{(m)}(t)\big)\,.$$

Since $\mu^\sharp_{(m)}(\cdot) \ge \zeta_{(m)}(\cdot)$ from (7.7) and the remark following it, the last term in (11.31) is monotonically increasing in T. It measures the “leakage” that occurs, when a capitalization-weighted portfolio is contained inside a larger market, and stocks cross-over (“leak”) from the cap-weighted to the market portfolio. For details of these derivations, see Fernholz (2002), pp. 84-88. □

Proof of the "Master Equation" (11.2). To ease notation we set

$$g_i(t) := D_i \log \mathbf{G}(\mu(t)) \qquad \text{and} \qquad N(t) := 1 - \sum_{j=1}^{n} \mu_j(t) g_j(t)\,,$$

so (11.1) reads: $\pi_i(t) = \big(g_i(t) + N(t)\big)\mu_i(t)$, $i = 1, \cdots n$. Then the terms on the right-hand side of (3.9) become

$$\sum_{i=1}^{n} \frac{\pi_i(t)}{\mu_i(t)}\, d\mu_i(t) \;=\; \sum_{i=1}^{n} g_i(t)\, d\mu_i(t) + N(t) \cdot d\Big(\sum_{i=1}^{n} \mu_i(t)\Big) \;=\; \sum_{i=1}^{n} g_i(t)\, d\mu_i(t)$$

and

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n} \pi_i(t)\pi_j(t)\tau_{ij}^{\mu}(t) \;&=\; \sum_{i=1}^{n}\sum_{j=1}^{n} \big(g_i(t) + N(t)\big)\big(g_j(t) + N(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}^{\mu}(t)\\ &=\; \sum_{i=1}^{n}\sum_{j=1}^{n} g_i(t)g_j(t)\mu_i(t)\mu_j(t)\tau_{ij}^{\mu}(t),\end{aligned}$$

the latter thanks to (1.21) and Lemma 3.1. Thus, (3.9) gives

$$d\log\left(\frac{V^{\pi}(t)}{V^{\mu}(t)}\right) \;=\; \sum_{i=1}^{n} g_i(t)\, d\mu_i(t) - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} g_i(t)g_j(t)\mu_i(t)\mu_j(t)\tau_{ij}^{\mu}(t)\, dt. \tag{11.32}$$

On the other hand, we have

$$D_{ij}^2 \log \mathbf{G}(x) \;=\; \big(D_{ij}^2 G(x)/G(x)\big) - D_i \log \mathbf{G}(x) \cdot D_j \log \mathbf{G}(x)\,,$$

so we get

$$\begin{aligned}d\log \mathbf{G}(\mu(t) &= \sum_{i=1}^{n} g_i(t)\, d\mu_i(t) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} D_{ij}^2 \log \mathbf{G}(\mu(t))\, d\langle \mu_i, \mu_j\rangle(t)\\ &= \sum_{i=1}^{n} g_i(t)\, d\mu_i(t) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \Big(\frac{D_{ij}^2 \mathbf{G}(\mu(t))}{\mathbf{G}(\mu(t))} - g_i(t)g_j(t)\Big)\mu_i(t)\mu_j(t)\tau_{ij}^{\mu}(t)\, dt\end{aligned}$$

by Itô's rule in conjunction with (2.6). Comparing this last expression with (11.32) and recalling (11.3), we deduce (11.2), namely $d\log \mathbf{G}(\mu(t) = d\log\left(V^{\pi}(t)/V^{\mu}(t)\right) - \mathfrak{g}(t)dt$. □

Chapter IV

Abstract Markets

The basic market model in (1.1) is too general for us to be able to draw many interesting conclusions. Hence, we would like to consider a more restricted class of models that still capture certain aspects of real equity markets, but are more analytically tractable than the general model (1.1). *Abstract markets* are relatively simple stochastic equity market models that exhibit selected characteristics of real equity markets, so that an understanding of these models will provide some insight into the behavior of actual markets. In particular, there are two classes of abstract markets that we shall discuss here: *volatility-stabilized markets* introduced in Fernholz & Karatzas (2005), and *rank-based models* exemplified by *Atlas models* and their generalizations, which first appeared in Fernholz (2002), with further development in Banner, Fernholz & Karatzas (2005).

12 Volatility-Stabilized Markets

Volatility-stabilized market models are remarkable, because in these models the market itself behaves in a rather sedate fashion, viz., (exponential) Brownian motion with drift, while the individual stocks are going all over the place (in a rigorously defined manner, of course). These models reflect the fact that in real markets, the smaller stocks tend to have greater volatility than the larger stocks.

Let us consider the abstract market model $\mathcal{M}$ with

$$d \log X_i(t) = \frac{\alpha}{2\mu_i(t)}\, dt + \frac{1}{\sqrt{\mu_i(t)}}\, dW_i(t), \qquad i = 1, \cdots, n, \tag{12.1}$$

where $\alpha \geq 0$ is a given real constant. The theory developed by Bass & Perkins (2002) shows that the resulting system of stochastic differential equations, for $i = 1, \ldots, n$,

$$dX_i(t) = \frac{1+\alpha}{2}\big(X_1(t) + \ldots + X_n(t)\big)\, dt + \sqrt{X_i(t)\big(X_1(t) + \ldots + X_n(t)\big)}\, dW_i(t), \tag{12.2}$$

determines the distribution of the Δ^n_+-valued diffusion process $\mathfrak{X}(\cdot) = \big(X_1(\cdot), \cdots, X_n(\cdot)\big)'$ uniquely; and that the conditions of (1.2), (6.4) are satisfied by the processes

$$b_i(\cdot) = (1+\alpha)/2\mu_i(\cdot)\,, \quad \sigma_{i\nu}(t) = (\mu_i(t))^{-1/2}\delta_{i\nu}\,, \quad r(\cdot) \equiv 0 \quad \text{and} \quad \theta_\nu(\cdot) = (1+\alpha)/2\sqrt{\mu_\nu(\cdot)}$$

for $1 \leq i, \nu \leq n$. The reader might wish to remark that condition (3.10) is satisfied in this case, in fact with $\varepsilon = 1$; but (1.16) fails.

The model of (12.1) assigns to all stocks log-drifts $\gamma_i(t) = \alpha/2\mu_i(t)$, covariances $a_{ij}(t) = 0$ for $j \neq i$, and variances $a_{ii}(t) = 1/\mu_i(t)$, $i = 1, \cdots, n$ that are largest for the smallest stocks and smallest for the largest stocks. Not surprisingly then, individual stocks fluctuate rather widely in a market of this type; in particular, *diversity fails* on every $[0, T]$; see Remarks 12.2 and 12.3.

Yet despite these fluctuations, the overall market has quite stable behavior. We call this phenomenon *stabilization by volatility* in the case $\alpha = 0$; and *stabilization by both volatility and drift* in the case $\alpha > 0$.

Indeed, the quantities $a_{\mu\mu}(\cdot)$, $\gamma^*_\mu(\cdot)$, $\gamma_\mu(\cdot)$ are computed from (1.20), (1.13), (1.12) as

$$a_{\mu\mu}(\cdot) \equiv 1, \quad \gamma^*_\mu(\cdot) \equiv \gamma^* := \frac{n-1}{2} > 0, \quad \gamma_\mu(\cdot) \equiv \gamma := \frac{(1+\alpha)n - 1}{2} > 0. \tag{12.3}$$

This, in conjunction with (2.2), computes the total market capitalization

$$X(t) \;=\; X_1(t) + \ldots + X_n(t) \;=\; X(0)\, e^{\gamma t + \mathcal{W}(t)}\,, \qquad 0 \leq t < \infty \tag{12.4}$$

as the exponential of the standard, one-dimensional Brownian motion $\mathcal{W}(\cdot) := \sum_{\nu=1}^n \int_0^\cdot \sqrt{\mu_\nu(s)}\, dW_\nu(s)$, plus drift $\gamma t > 0$. In particular, the overall market and the largest stock $X_{(1)}(\cdot) = \max_{1 \leq i \leq n} X_i(\cdot)$ grow at the same, constant rate:

$$\lim_{T\to\infty} \frac{1}{T} \log X(T) = \lim_{T\to\infty} \frac{1}{T} \log X_{(1)}(T) = \gamma, \quad \text{a.s.} \tag{12.5}$$

On the other hand, according to Example 11.1 there exist in this model portfolios that lead to strong arbitrage opportunities relative to the market, at least on time horizons $[0, T]$ with $T \in (\mathcal{T}_*, \infty)$, where

$$\mathcal{T}_* := \frac{2\,\mathbf{H}(\mu(0))}{n-1} \;\leq\; \frac{2 \log n}{n-1}\,. \tag{12.6}$$

To wit: *strong relative arbitrage can exist in non-diverse markets with unbounded volatilities.*

The last upper-bound in the above expression (12.6) becomes small as the number of stocks in the market increases. In fact, Banner & Fernholz (2007) provided recently an elaborate construction which shows that strong arbitrage exists, relative to the market described by (12.1), over *arbitrary* time-horizons.

12.1 Bessel Processes

The crucial observation now, is that the solution of the system (12.1) can be expressed in terms of the squares of independent Bessel processes $\mathfrak{R}_1(\cdot), \ldots, \mathfrak{R}_n(\cdot)$ in dimension $\kappa := 2(1+\alpha) \geq 2$, and of an appropriate time change:

$$X_i(t) = \mathfrak{R}_i^2\big(\Lambda(t)\big), \qquad 0 \leq t < \infty, \quad i = 1, \ldots, n, \tag{12.7}$$

where

$$\Lambda(t) := \frac{1}{4} \int_0^t X(u)\, du = \frac{X(0)}{4} \int_0^t e^{\gamma s + \mathcal{W}(s)}\, ds, \qquad 0 \leq t < \infty \tag{12.8}$$

and

$$\mathfrak{R}_i(u) = \sqrt{X_i(0)} + \frac{\kappa - 1}{2} \int_0^u \frac{d\xi}{\mathfrak{R}_i(\xi)} + \mathfrak{W}_i(u), \qquad 0 \leq u < \infty. \tag{12.9}$$

Here, the driving processes $\mathfrak{W}_i(\cdot) := \int_0^{\Lambda^{-1}(\cdot)} \sqrt{\Lambda'(t)}\, dW_i(t)$ are independent, standard one-dimensional Brownian motions (e.g. [KS] (1991), pp. 157-162). In a similar vein, we have the representation

$$X(t) = \mathfrak{R}^2\big(\Lambda(t)\big), \quad 0 \leq t < \infty$$

of the total market capitalization, in terms of the Bessel process

$$\mathfrak{R}(u) = \sqrt{X(0)} + \frac{n\kappa - 1}{2} \int_0^u \frac{d\xi}{\mathfrak{R}(\xi)} + \mathfrak{W}(u), \qquad 0 \leq u < \infty \tag{12.10}$$

in dimension $n\kappa$, and of yet another one-dimensional Brownian motion $\mathfrak{W}(\cdot)$.

This observation provides a wealth of structure, which can be used then to study the asymptotic properties of the model (12.1).

Remark 12.1. For the case $\alpha > 0$ ($\kappa > 2$), we have the ergodic property

$$\lim_{u \to \infty} \frac{1}{\log u} \int_0^u \frac{d\xi}{\mathfrak{R}_i^2(\xi)} = \frac{1}{\kappa - 2} = \frac{1}{2\alpha}, \qquad \text{a.s.}$$

(a consequence of the Birkhoff ergodic theorem and of the strong Markov property of the Bessel process), as well as the *Lamperti representation*

$$\mathfrak{R}_i(u) = \sqrt{x_i}\, e^{\alpha\theta + \mathfrak{B}_i(\theta)} \Big|_{\theta = \int_0^u \mathfrak{R}_i^{-2}(\xi) d\xi}, \qquad 0 \leq u < \infty$$

for the Bessel process $\mathfrak{R}_i(\cdot)$ in terms of the exponential of a standard Brownian motion $\mathfrak{B}_i(\cdot)$ with positive drift $\alpha > 0$. From these considerations, one can deduce the a.s. properties

$$\lim_{u \to \infty} \frac{\log \mathfrak{R}_i(u)}{\log u} = \frac{1}{2}, \qquad \lim_{t \to \infty} \frac{1}{t} \log X_i(t) = \gamma, \tag{12.11}$$

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T a_{ii}(t)\, dt = \lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{dt}{\mu_i(t)} = \frac{2\gamma}{\alpha} = n + \frac{n-1}{\alpha}, \tag{12.12}$$

for each $i = 1, \ldots, n$; see pp. 174-175 in [FK] (2005) for details. In particular, all stocks grow at the same asymptotic rate $\gamma > 0$ of (12.3), as does the entire market; the model of (12.1) is coherent in the sense of Remark 2.1; and the conditions (1.6), (1.7) hold. $\square$

Remark 12.2. In the case $\alpha = 0$ ($\kappa = 2$), it can be shown that

$$\lim_{u \to \infty} \frac{\log \mathfrak{R}_i(u)}{\log u} = \frac{1}{2} \qquad \text{holds in probability}, \tag{12.13}$$

but that we have almost surely:

$$\limsup_{u\to\infty} \frac{\log \mathfrak{R}_i(u)}{\log u} = \frac{1}{2}\,, \qquad \liminf_{u\to\infty} \frac{\log \mathfrak{R}_i(u)}{\log u} = -\infty. \tag{12.14}$$

It follows from this and (12.5) that

$$\lim_{t\to\infty} \frac{1}{t} \log X_i(t) = \gamma \qquad \text{holds in probability,} \tag{12.15}$$

and also that

$$\limsup_{t\to\infty} \frac{1}{t} \log X_i(t) = \gamma\,, \qquad \liminf_{t\to\infty} \frac{1}{t} \log X_i(t) = -\infty \tag{12.16}$$

hold almost surely. To wit, individual stocks can "crash" in this case, despite the overall stability of the market; and coherence now fails, as does the condition (1.6).

(*Note:* The claim (12.13) comes form the observation

$$\mathfrak{R}_i(u) \;=\; ||\,\mathfrak{R}_i(0) + \mathfrak{b}_i(u)\,|| \;=\; \sqrt{u}\;\; ||\,(\mathfrak{R}_i(0)/\sqrt{u}\,) + \mathfrak{b}_i(1)\,|| \quad \text{in distribution,}$$

where $\mathfrak{R}_i(\cdot)$ and $\mathfrak{b}_i(\cdot)$ are Brownian motions on the plane and on the real line, respectively; thus, we have $\lim_{u\to\infty} \big(\log \mathfrak{R}_i(u) - (1/2)\log u\big) = \log ||\mathfrak{b}_i(1)||$ in distribution, and (12.13) follows.

As for (12.14), its fist claim follows from the law of the iterated logarithm for Brownian motion on the real line; whereas the second claim is obtained from the following result:

For a decreasing function $h(\cdot)$ we have

$$\mathbb{P}\left(\,\mathfrak{R}_i(u) \geq u^{1/2} h(u) \quad \text{for all} \;\; u > 0 \;\text{ sufficiently large}\right) = \;1 \;\text{ or }\; 0\,,$$

depending on whether the series $\sum_{k\in\mathbb{N}} \big(k\,|\log h(k)\,|\big)^{-1}$ *converges or diverges.*

This zero-one law is due to Spitzer (1958); details of the argument can be found on pp. 176-177 of [FK] (2005).) □

Remark 12.3. In the case $\alpha = 0$ ($\kappa = 2$), it can be shown that

$$\lim_{u\to\infty} \mathbb{P}\big(\mu_i\big(\Lambda^{-1}(u)\big) > 1-\delta\big) = \delta^{n-1}$$

holds for every $i = 1,\ldots,n$ and $\delta \in (0,1)$; here $\Lambda^{-1}(\cdot) = 4\int_0^{\cdot} \mathfrak{R}^{-2}(\xi)\,d\xi$ is the inverse of the time change $\Lambda(\cdot)$ in (12.8), and $\mathfrak{R}(\cdot)$ is the Bessel process in (12.10). It follows that this model is not diverse on $[0,\infty)$. □

Remark 12.4. The exponential strict local martingale of (6.5) can be computed as

$$Z(T) \;=\; \exp\left\{\, \frac{\alpha^2-1}{8} \sum_{i=1}^{n} \int_0^T \frac{X_1(t)+\cdots+X_n(t)}{X_i(t)}\; dt \right\} \cdot \left(\frac{X_1(0)\cdots X_n(0)}{X_1(T)\cdots X_n(T)} \right)^{(1+\alpha)/2} .$$

Thus, the log-optimal trading strategy $h_*(\cdot)$ and its associated wealth process $V_*(\cdot) \equiv \mathcal{V}^{1,h^*}(\cdot)$ of Remark 10.5, are given as $V_*(\cdot) = 1/Z(\cdot)$ and $h_i^*(\cdot) = (1+\alpha)V_*(\cdot)/2\,,\; i = 1,\cdots,n\,.$

For $\alpha > 0$, we deduce from this and (12.11), (12.12) that we have the following a.s. growth rates:

$$\lim_{T\to\infty} \frac{1}{T} \log V_*(T) \;=\; n\gamma(1+\alpha)^2/4\alpha$$

and therefore

$$\lim_{T\to\infty} \frac{1}{T} \log\left(\frac{V_*(T)}{V^{\mu}(T)}\right) \;=\; \left(\frac{n(1+\alpha)^2}{4\alpha} - 1\right)\gamma \;=\; \left(\frac{n(1+\alpha)^2}{4\alpha} - 1\right) \frac{(1+\alpha)n-1}{2}\,. \tag{12.17}$$

□

Example 12.1. Diversity Weighting: In the context of the volatility-stabilized model of this section with $p = 1/2$, the diversity-weighted portfolio

$$\mu_i^{(p)}(t) = \frac{\sqrt{\mu_i(t)}}{\sum_{j=1}^n \sqrt{\mu_j(t)}}\,, \qquad i = 1, \cdots, n$$

of (7.1) represents a strong arbitrage relative to the market portfolio, namely

$$\mathbb{P}\left[V^{\pi^{(p)}}(T) > V^{\mu}(T) \right] = 1\,, \quad \text{at least on time-horizons } [0,T] \text{ with } T > \frac{8 \log n}{n-1}\,.$$

Furthermore, this diversity-weighted portfolio outperforms considerably the market over long time-horizons:

$$\mathcal{L}^{\mu^{(p)},\mu} := \liminf_{T\to\infty} \frac{1}{T} \log\left(\frac{V^{\mu^{(p)}}(T)}{V^{\mu}(T)} \right) = \liminf_{T\to\infty} \frac{1}{2T} \int_0^T \gamma^*_{\mu^{(p)}}(t)\, dt \geq \frac{n-1}{8}\,, \qquad \text{a.s.}$$

Question: Do the indicated limits exist? Can they be computed in closed form? □

Example 12.2. Equal Weighting: With a covariance structure of the form $a_{ij}(t) = \big(1/\mu_i(t)\big)\,\delta_{ij}$, as in the volatility-stabilized model of the present section, the excess growth rate $\gamma^*_{\varphi}(\cdot)$ in (7.11) for the equally weighted portfolio $\varphi(\cdot)$ of Remark 7.2 takes the form

$$\gamma^*_{\varphi}(\cdot) = \frac{n-1}{2n^2} \sum_{i=1}^n \frac{1}{\mu_i(t)}\,.$$

The geometric-mean/harmonic-mean inequality now implies that the condition (11.16) is satisfied by the constant $\zeta = (n-1)/2n$; thus, according to Example 11.2, the market-modulated, equally weighted portfolio $\varphi^c(\cdot)$ of (11.4) is a strong arbitrage opportunity relative to the market, over time-horizons $[0,T]$ with $T > 2\,n^{1-(1/n)}/(n-1)$, provided that $c > 0$ is chosen sufficiently large in (11.4).

How much better is equal weighting, relative to the volatility-stabilized market of this section with $\alpha > 0$, over *very large* time-horizons? In conjunction with (7.10) and the coherence property of this market, the strong law of large numbers (12.12) implies that the limit

$$\mathcal{L}^{\varphi,\mu} := \lim_{T\to\infty} \frac{1}{T} \log\left(\frac{V^{\varphi}(T)}{V^{\mu}(T)} \right) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \gamma^*_{\varphi}(t)\, dt$$

of (6.2) exists a.s., and equals

$$\mathcal{L}^{\varphi,\mu} = \frac{n-1}{2} \left(1 + \frac{n-1}{n\alpha} \right). \tag{12.18}$$

In other words: equal weighting, with its built-in "buying low and selling high" features, outperforms *considerably* this drift- and volatility-stabilized market, over long time-horizons. □

Example 12.3. Growth Optimality: For the volatility-stabilized model of this section with $0 < \alpha < 1$ and $\lambda := \gamma + (1/2) = n(1+\alpha)/2 \geq 1$, the portfolio

$$\widehat{\pi}_i(t) := \frac{1+\alpha}{2} - \Big(\frac{n}{2}\,(1+\alpha) - 1 \Big)\, \mu_i(t) = \lambda\varphi_i(t) - (\lambda - 1)\mu_i(t)\,, \qquad i = 1, \cdots, n \tag{12.19}$$

maximizes pointwise the growth rate as in (4.1) of Problem 4.6, section 4: it is the growth-optimal portfolio for this model. Its excess growth rate is computed as

$$\gamma^*_{\widehat{\pi}}(t) = \frac{\lambda(n-\lambda)}{2n^2} \sum_{i=1}^n \frac{1}{\mu_i(t)} - \frac{\lambda - 1}{2}\,\big(n - \lambda - 1\big)\,.$$

Note that $\widehat{\pi}(\cdot)$ is long in the equally weighted portfolio $\varphi(\cdot)$ of Example 12.2, and short in the market portfolio $\mu(\cdot)$. Using the structure of these two simple portfolios, it is relatively straightforward to compute the performance of $\widehat{\pi}(\cdot)$ relative to the market, namely

$$\log\left(\frac{V^{\widehat{\pi}}(T)}{V^{\mu}(T)}\right) = \frac{\lambda}{n}\log\left(\frac{\mu_1(T)\cdots\mu_n(T)}{\mu_1(0)\cdots\mu_n(0)}\right) + \int_0^T \Big(\gamma^*_{\widehat{\pi}}(t) + (\lambda-1)\,\gamma^*_{\mu}(t)\Big)\,dt\,.$$

Recalling the coherence of this model, the asymptotic property (12.12), and the computation $\gamma^*_\mu(t) = (n-1)/2$, we deduce

$$\mathcal{L}^{\widehat{\pi},\mu} := \lim_{T\to\infty}\frac{1}{T}\log\left(\frac{V^{\widehat{\pi}}(T)}{V^{\mu}(T)}\right) = \frac{\lambda(n-\lambda)}{n}\cdot\frac{\gamma}{\alpha} + \frac{\lambda(\lambda-1)}{2} \tag{12.20}$$

$$= \frac{n^2}{8}(1+\alpha)\left[1+\alpha+\frac{1}{2n}+(1-\alpha)\left(1+\frac{n-1}{\alpha n}\right)\right].$$

A comparison with (12.18) shows that shorting the market portfolio as in (12.19), improves the performance of equal weighting by an entire order of magnitude in terms of market-size n; whilst the quantity of (12.20) is smaller than that of (12.17), as of course it should be, but has the same order of magnitude in terms of market-size. □

Remark 12.5. Open Question: For the entropy-weighted portfolio $\pi^c_i(\cdot)$ of (11.10), compute in the context of the volatility-stabilized model the expression

$$\mathcal{L}^{\pi^c,\mu} := \liminf_{T\to\infty}\frac{1}{T}\log\left(\frac{V^{\pi^c}(T)}{V^{\mu}(T)}\right) = \liminf_{T\to\infty}\frac{\gamma^*}{T}\int_0^T\frac{dt}{c+\mathbf{H}(\mu(t))}$$

of (6.2), using (11.10) and (12.3). But note already from these expressions that

$$\mathcal{L}^{\pi^c,\mu} \geq \frac{n-1}{2(c+\log n)} > 0 \quad \text{a.s.},$$

suggesting again a significant outperforming of the market over long time-horizons. Do the indicated limits exist, as one would expect? □

Remark 12.6. Open Questions: For fixed $t \in (0,\infty)$, determine the distributions of $\mu_i(t)$, $i = 1,\cdots,n$ and of the largest $\mu_{(1)}(t) := \max_{1\le i\le n}\mu_i(t)$ and smallest $\mu_{(n)}(t) := \min_{1\le i\le n}\mu_i(t)$ market weights.

What can be said about the behavior of the averages $\frac{1}{T}\int_0^T \mu_{(k)}(t)dt$, particularly for the largest $(k=1)$ and the smallest $(k=n)$ stocks? □

13 Rank-Based Models

Size is one of the most important descriptive characteristics of financial assets. One can understand a lot about equity markets by observing, and trying to make sense of, the continual ebb and flow of small-, medium- and large-capitalization stocks in their midst. A particularly convenient way to study this feature is by looking at the evolution of the *capital distribution curve* $\log k \mapsto \log \mu_{(k)}(t)$; that is, the logarithms of the market weights arranged in descending order, versus the logarithms of their respective ranks (see also (13.14) below for a steady-state counterpart of this quantity). As shown in Figure 5.1 of Fernholz (2002), reproduced here as Figure 13.4, this log-log plot has exhibited remarkable stability over the decades of the last century.

It is of considerable importance, then, to have available models which describe this flow of capital and exhibit stability properties for capital distribution that are in at least broad agreement with these observations.

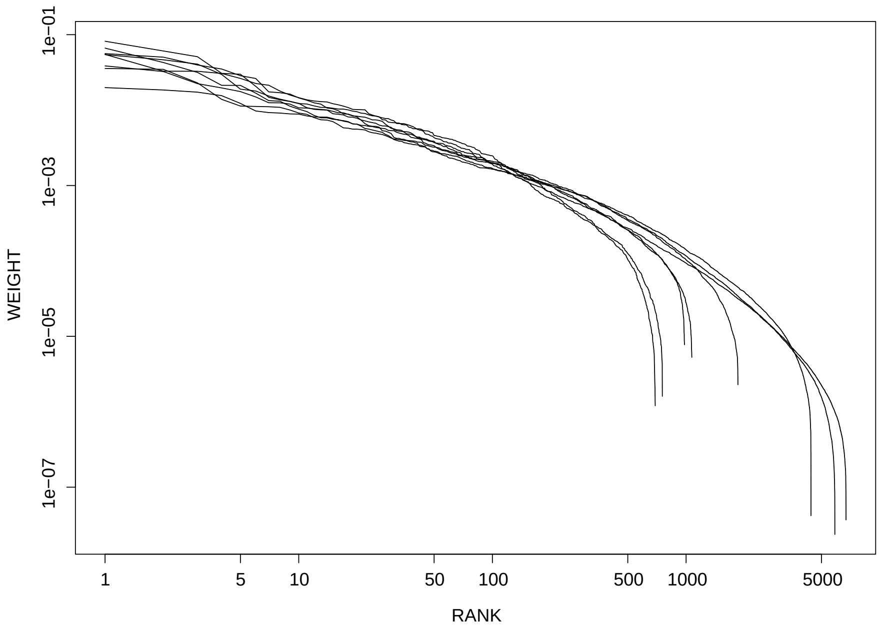

Figure 13.4: Capital distribution curves: 1929–1999.
The later the period, the longer the curve.

The simplest model of this type assigns growth rates and volatilities to the various stocks, not according to their names (the indices i) but according to their *ranks* within the market's capitalization. More precisely, let us pick real numbers $\gamma, g_1, \ldots, g_n$ and $\sigma_1 > 0, \ldots, \sigma_n > 0$, satisfying conditions that will be specified in a moment, and prescribe growth rates $\gamma_i(\cdot)$ and volatilities $\sigma_{i\nu}(\cdot)$

$$\gamma_i(t) \;=\; \gamma + \sum_{k=1}^{n} g_k 1_{\{X_i(t)=X_{p_t(k)}(t)\}} \qquad \sigma_{i\nu}(t) \;=\; \delta_{i\nu} \cdot \sum_{k=1}^{n} \sigma_k \, 1_{\{X_i(t)=X_{p_t(k)}(t)\}} \tag{13.1}$$

for $1 \le i, \nu \le n$ with $d = n$. We are using here the random permutation notation of (11.18), and we shall denote again by $\mathfrak{X}(\cdot) = \big(X_1(\cdot), \cdots, X_n(\cdot)\big)'$ the vector of stock-capitalizations.

It is intuitively clear that if such a model is to have some stability properties, it has to assign considerably higher growth rates to the smallest stocks than to the biggest ones. It turns out that the right conditions for stability are

$$g_1 < 0\,, \quad g_1 + g_2 < 0\,, \quad \ldots \quad , \quad g_1 + \cdots + g_{n-1} < 0\,, \quad g_1 + \cdots + g_n = 0\,. \tag{13.2}$$

These conditions are satisfied in the simplest model of this type, the *Atlas model* that assigns

$$\gamma = g > 0\,, \quad g_k = -g \ \text{ for } \ k = 1, \ldots, n-1 \quad \text{and} \quad g_n = (n-1)g\,, \tag{13.3}$$

thus $\gamma_i(t) = ng \, 1_{\{X_i(t)=X_{p_t(n)}(t)\}}$ in (13.1): zero growth rate goes to all the stocks but the smallest, which then becomes responsible for supporting the entire growth of the market.

In addition to the drift condition (13.2), we shall impose a condition on the variances of the model:

$$\sum_{k=1}^{n} \sigma_k^2 \;>\; 2 \cdot \max_{1 \le k \le n} \sigma_k^2\,, \qquad 0 \le \sigma_2^2 - \sigma_1^2 \le \sigma_3^2 - \sigma_2^2 \le \cdots \le \sigma_n^2 - \sigma_{n-1}^2\,.$$

Making these specifications amounts to postulating that the log-capitalizations $Y_i(\cdot) := \log X_i(\cdot)$ $i = 1, \cdots, n$ satisfy the system of stochastic differential equations

$$dY_i(t) \;=\; \Big(\gamma + \sum_{k=1}^{n} g_k 1_{\mathcal{Q}_i^{(k)}}(\mathfrak{Y}(t))\Big)dt \;+\; \sum_{k=1}^{n} \sigma_k 1_{\mathcal{Q}_i^{(k)}}(\mathfrak{Y}(t))\, dW_i(t)\,, \tag{13.4}$$

with $Y_i(0) = y_i = \log x_i$. Here $\{\mathcal{Q}_i^{(k)}\}_{1 \le i,k \le n}$ is a collection of polyhedral domains in $\mathbb{R}^n$, with the properties

$$\{\mathcal{Q}_i^{(k)}\}_{1 \le i \le n} \quad \text{is a partition of } \mathbb{R}^n, \quad \text{for each fixed } k\,,$$
$$\{\mathcal{Q}_i^{(k)}\}_{1 \le k \le n} \quad \text{is a partition of } \mathbb{R}^n, \quad \text{for each fixed } i\,,$$

and the interpretation

$$\mathfrak{Y} = (Y_1, \ldots, Y_n) \in \mathcal{Q}_i^{(k)} \quad \text{means that } \ Y_i \ \text{ is ranked } \ k^{\text{th}} \text{ among } \ Y_1, \ldots, Y_n\,.$$

As long as the vector of log-capitalizations $\mathfrak{Y}(\cdot) = \big(Y_1(\cdot), \cdots, Y_n(\cdot)\big)'$ is in the polyhedron $\mathcal{Q}_i^{(k)}$, the equation (13.3) posits that the coördinate process $Y_i(\cdot)$ evolves like a Brownian motion with drift $\gamma + g_k$ and variance σ_k^2. (Ties are resolved by resorting to the lowest index i; for instance, $\mathcal{Q}_i^{(1)}$, $1 \le i \le n$ corresponds to the partition $\mathcal{Q}_i$ of $(0,\infty)^n$ of section 9, right below (9.3); and so on.)

The theory of Bass & Pardoux (1987) guarantees that this system has a weak solution, which is unique in distribution; once this solution has been constructed, we obtain stock capitalizations as $X_i(\cdot) = e^{Y_i(\cdot)}$ that satisfy (1.4) with the specifications of (13.1).

Remark 13.1. Research Problem: There is a natural generalization of (13.4) to

$$dY_i(t) \;=\; \Big(\gamma_i + \sum_{k=1}^{n} g_k 1_{\mathcal{Q}_i^{(k)}}(\mathfrak{Y}(t))\Big)dt \;+\; \sum_{k=1}^{n} \sigma_k 1_{\mathcal{Q}_i^{(k)}}(\mathfrak{Y}(t))\, dW_i(t) + \rho_i\, dB_i(t), \tag{13.5}$$

where $(B_1(\cdot), \ldots, B_n(\cdot))$ is a Brownian motion independent of $(W_1(\cdot), \ldots, W_n(\cdot))$, and the γ_i and ρ_i are constants. In this case, it can be shown that the system is stable if and only if, besides (13.2), we have $\gamma_1 + \cdots + \gamma_n = 0$ and

$$\sum_{k=1}^{\ell} \big(g_k + \gamma_{\pi(k)}\big) \;<\; 0\,, \qquad \ell = 1, \cdots, n-1\,,$$

for any permutation π of $\{1, 2, \ldots, n\}$. The model (13.5) is known as the *hybrid model*, since the growth rates and variances depend of both rank and name, i.e., index. These models provide a simplification of the general market model of (1.1), but nevertheless one that may be both tractable enough and ample enough to allow meaningful insight into the behavior of real equity markets. Be that as it may, at this writing, there remain many open research questions regarding these hybrid models. □

An immediate observation from (13.3) is that the sum $Y(\cdot) := \sum_{i=1}^n Y_i(\cdot)$ of log-capitalizations satisfies

$$Y(t) \;=\; y + n\gamma t + \sum_{k=1}^{n} \sigma_k B_k(t)\,, \qquad 0 \le t < \infty$$

with $y := \sum_{i=1}^n y_i$, and $B_k(\cdot) := \sum_{i=1}^n \int_0^\cdot 1_{\mathcal{Q}_i^{(k)}}(\mathfrak{Y}(s))dW_i(s)$, $k = 1, \ldots n$ independent scalar Brownian motions. Thus, the strong law of large numbers implies

$$\lim_{T\to\infty} \frac{1}{T} \sum_{i=1}^{n} Y_i(T) = n\gamma\,, \qquad \text{a.s.}$$

Then it takes a considerable amount of work (see Appendix in [BFK] 2005), in order to strengthen this result to

$$\lim_{T\to\infty} \frac{1}{T} \log X_i(T) = \lim_{T\to\infty} \frac{Y_i(T)}{T} = \gamma \quad \text{a.s., for every } \ i = 1, \ldots, n\,; \tag{13.6}$$

to wit, all the stocks have the same asymptotic growth-rate γ in this model. Using (13.6), it can be shown that the model specified by (1.5), (13.1) is coherent in the sense of Remark 2.1.

Remark 13.2. Taking Turns in the Various Ranks. From (13.4), (13.6) and the strong law of large numbers for Brownian motion, we deduce that the quantity $\sum_{k=1}^n g_k \left(\frac{1}{T} \int_0^T 1_{\mathcal{Q}_i^{(k)}}(\mathfrak{Y}(t))\, dt \right)$ converges a.s. to zero, as $T \to \infty$. For the Atlas model in (13.3), this expression becomes $g \left(\frac{n}{T} \int_0^T 1_{\mathcal{Q}_i^{(n)}}(\mathfrak{Y}(t))\, dt - 1 \right)$, and we obtain

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T 1_{\mathcal{Q}_i^{(n)}}(\mathfrak{Y}(t))\, dt \;=\; \frac{1}{n} \quad \text{a.s., for every} \;\; i = 1, \ldots, n.$$

Namely, each stock spends roughly $(1/n)^{\text{th}}$ of the time, acting as "Atlas".

Again with considerable work, this is strengthened in [BFK] (2005) to the statement

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T 1_{\mathcal{Q}_i^{(k)}}\big(\mathfrak{Y}(t)\big)\, dt \;=\; \frac{1}{n}, \qquad \text{a.s., for every} \;\; 1 \le i, k \le n, \tag{13.7}$$

valid not just for the Atlas model, but under the more general conditions of (13.2). Thanks to the symmetry inherent in this model, each stock spends roughly $(1/n)^{\text{th}}$ of the time in any given rank; see Proposition 2.3 in [BFK] (2005). □

13.1 Ranked Capitalization Processes

For many purposes in the study of these models, it makes sense to look at the ranked log-capitalization processes

$$Z_k(t) := \sum_{i=1}^n Y_i(t) \cdot 1_{\mathcal{Q}_i^{(k)}}(\mathfrak{Y}(t)), \quad 0 \le t < \infty \tag{13.8}$$

for $1 \le k \le n$. From these, we get the ranked capitalizations via $X_{(k)}(t) = e^{Z_k(t)}$, with notation similar to (1.18). Using an extended Tanaka-type formula, as we did in (11.19), it can be seen that the processes of (13.8) satisfy

$$Z_k(t) \;=\; Z_k(0) + (g_k + \gamma)t + \sigma_k B_k(t) + \frac{1}{2}\Big(\mathfrak{L}^{k,k+1}(t) - \mathfrak{L}^{k-1,k}(t) \Big), \quad 0 \le t < \infty \tag{13.9}$$

in that notation. Here, as in subsection 11.2, the continuous and increasing process $\mathfrak{L}^{k,k+1}(\cdot) := \Lambda_{\Xi_k}(\cdot)$ is the semimartingale local time at the origin of the continuous, non-negative process

$$\Xi_k(\cdot) \;=\; Z_k(\cdot) - Z_{k+1}(\cdot) \;=\; \log\big(\mu_{(k)}(\cdot)/\mu_{(k+1)}(\cdot) \big)$$

of (11.20) for $k = 1, \cdots, n-1$; and we make again the convention $\mathfrak{L}^{0,1}(\cdot) \equiv \mathfrak{L}^{n,n+1}(\cdot) \equiv 0$.

These local times play a big rôle in the analysis of this model. The quantity $\mathfrak{L}^{k,k+1}(T)$ represents again the cumulative amount of change between ranks k and $k+1$ that occurs over the time interval $[0, T]$. Of course, in a model such as the one studied here, the intensity of changes for the smaller stocks should be higher than for the larger stocks.

This is borne out by experiment: as we saw in Remark 11.8 it turns out, somewhat surprisingly, that these local times can be estimated based only on observations of relative market weights and of the performance of simple portfolios over $[0, T]$; and that they exhibit a remarkably linear increase, with positive rates that grow with k, as we see in Figure 13.5, reproduced from Fernholz (2002), Figure 5.2.

The analysis of the present model agrees with these observations: it follows from (13.6) and the dynamics of (13.9) that, for $k = 1, \ldots, n-1$, we have

$$\lim_{T\to\infty} \frac{1}{T}\, \mathfrak{L}^{k,k+1}(T) \;=\; \lambda_{k,k+1} \;:=\; -2\big(g_1 + \ldots + g_k\big) > 0, \qquad \text{a.s.} \tag{13.10}$$

Our stability condition guarantees that these partial sums are positive – as indeed the limits on the right-hand side of (13.10) ought to be; and in typical examples, such as the Atlas model of (13.3) where $\lambda_{k,k+1} = kg$, they do increase with k, as suggested by Figure 13.5.

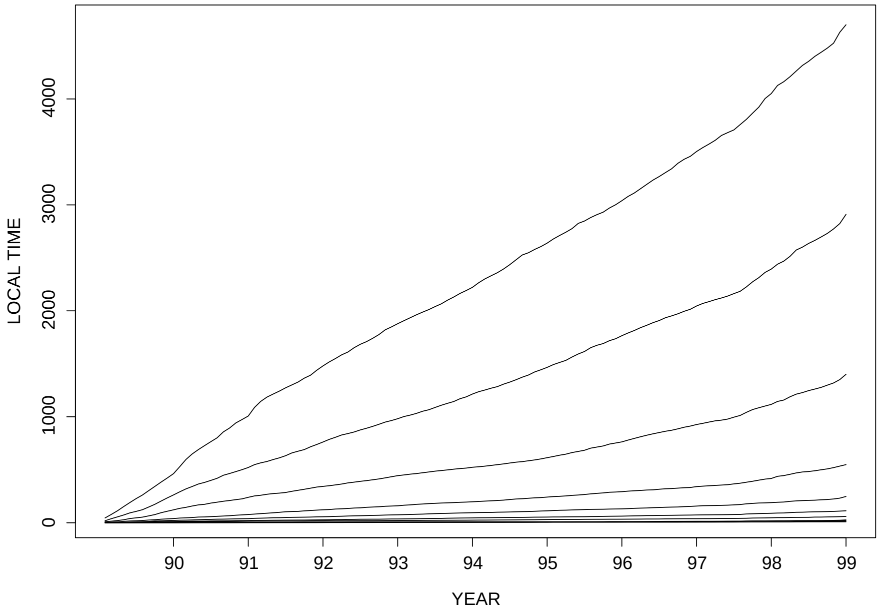

Figure 13.5: $\mathfrak{L}^{k,k+1}(\cdot)$, $k = 10, 20, 40, \cdots, 5120$.

13.2 Some Asymptotics

A slightly more careful analysis of these local times reveals that the non-negative semimartingale $\Xi_k(\cdot)$ of (11.20) can be cast in the form of a *Skorohod problem*

$$\Xi_k(t) = \Xi_k(0) + \Theta_k(t) + \Lambda_{\Xi_k}(t), \qquad 0 \le t < \infty,$$

as the reflection, at the origin, of the semimartingale

$$\Theta_k(t) = (g_k - g_{k+1})\, t - \frac{1}{2}\Big(\mathfrak{L}^{k-1,k}(t) + \mathfrak{L}^{k+1,k+2}(t)\Big) + \mathbf{s}_k \widetilde{W}^{(k)}(t),$$

where $\mathbf{s}_k := \big(\sigma_k^2 + \sigma_{k+1}^2\big)^{1/2}$ and $\widetilde{W}^{(k)}(\cdot) := \big(\sigma_k B_k(\cdot) - \sigma_{k+1} B_{k+1}(\cdot)\big)/\mathbf{s}_k$ is standard Brownian Motion.

As a result of these observations and of (13.10), we conclude that the process $\Xi_k(\cdot)$ behaves asymptotically like Brownian motion with drift $g_k - g_{k+1} - \frac{1}{2}\,(\lambda_{k-1,k} + \lambda_{k,k+1}) = -\lambda_{k,k+1} < 0$, variance $\mathbf{s}_k^2$, and reflection at the origin. Consequently,

$$\lim_{t\to\infty} \log\left(\frac{\mu_{(k)}(t)}{\mu_{(k+1)}(t)}\right) = \lim_{t\to\infty} \Xi_k(t) = \xi_k\,, \quad \text{in distribution} \tag{13.11}$$

where, for each $k = 1, \ldots, n-1$ the random variable ξ_k has an exponential distribution

$$\mathbb{P}(\xi_k > x) = e^{-r_k x},\ x \ge 0 \quad \text{with parameter} \quad r_k := \frac{2\lambda_{k,k+1}}{\mathbf{s}_k^2} = -\frac{4(g_1 + \cdots + g_k)}{\sigma_k^2 + \sigma_{k+1}^2} > 0\,. \tag{13.12}$$

As T. Ichiba (2006) observes, the theory of Harrison & Williams (1987)(a,b) implies that the random variables $\xi_1, \cdots, \xi_n$ are *independent* when the variances are of the form $\sigma_k^2 = \sigma^2 + ks^2$ for some real numbers $\sigma^2 > 0$ and $s^2 \ge 0$; that is, are either constant, or grow linearly with rank.

13.3 The Steady-State Capital Distribution Curve

We also have from (13.11) the strong law of large numbers

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T g\big(\Xi_k(t)\big)\,dt=\mathbb{E}\big(g(\xi_k)\big),\qquad \text{a.s.}$$

for every rank k, and every measurable function $g:[0,\infty)\to\mathbb{R}$ with $\int_0^\infty |g(x)|e^{-r_k x}dx<\infty$; see Khas'minskii (1960). In particular,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \log\left(\frac{\mu_{(k)}(t)}{\mu_{(k+1)}(t)}\right)dt\;=\;\mathbb{E}\big(\xi_k\big)\;=\;\frac{1}{r_k}\;=\;\frac{\mathbf{s}_k^2}{2\lambda_{k,k+1}},\qquad \text{a.s.}\tag{13.13}$$

This observation provides a tool for studying the *steady-state capital distribution curve*

$$\log k\;\longmapsto\;\lim_{T\to\infty}\frac{1}{T}\int_0^T \log\mu_{(k)}(t)\,dt=:\mathfrak{m}(k),\quad k=1,\cdots,n-1\tag{13.14}$$

alluded to at the beginning of this section (more on the existence of this limit in the next subsection). To estimate the slope $\mathfrak{q}(k)$ of this curve at the point $\log k$, we use (13.13) and the estimate $\log(k+1)-\log k\approx 1/k$, to obtain in the notation of (13.12):

$$\mathfrak{q}(k)\;\approx\;\frac{\mathfrak{m}(k)-\mathfrak{m}(k+1)}{\log k-\log(k+1)}=-\frac{k}{r_k}=\frac{k\big(\sigma_k^2+\sigma_{k+1}^2\big)}{4(g_1+\cdots+g_k)}<0\,.\tag{13.15}$$

Consider now an Atlas model as in (13.3). With equal variances $\sigma_k^2=\sigma^2>0$, this slope is the constant $\mathfrak{q}(k)\approx-\sigma^2/2g$, and the steady-state capital distribution curve can be approximated by a straight *Pareto* line.

On the other hand, with variances of the form $\sigma_k^2=\sigma^2+ks^2$ for some $s^2>0$, growing linearly with rank, we get for large k the approximate slope

$$\mathfrak{q}(k)\approx-\frac{1}{2g}\big(\sigma^2+ks^2\big),\qquad k=1\,,\cdots,n-1\,.$$

Such linear growth is suggested by Figure 5.5 in Fernholz (2002), which is reproduced here as Figure 13.6. This would imply a *decreasing and concave* steady-state capital distribution curve, whose (negative) slope becomes more and more pronounced in magnitude with increasing rank, much in accord with the features of Figure 13.4.

We see, in other words, that even such a simplistic model as that of (1.5), (13.1) – which has features such as (13.6), (13.7) that are not particularly realistic – is able to capture asymptotic stability properties observed in real markets, such as those exhibited in Figures 4, 5 and 6. It is possible to modify the model of the present section in ways that remove the 'simplistic' features (13.6), (13.7), while at the same time retaining the good asymptotic properties already mentioned. One is thus led to the "hybrid" models of Remark 13.1, that prescribe growth rates and covariances based on both name (the index i) and rank; as already mentioned, such models are the subject of very active current research.

Remark 13.3. Estimation of Parameters in this Model. Let us remark that (13.10) provides a method for obtaining estimates $\widehat{\lambda}_{k,k+1}$ of the parameters $\lambda_{k,k+1}$, from the observable random variables $\mathfrak{L}^{k,k+1}(T)$ that measure cumulative change between ranks k and $k+1$; recall Remark 11.8 once again. Then estimates of the parameters g_k follow, as $\widehat{g}_k=\big(\widehat{\lambda}_{k-1,k}-\widehat{\lambda}_{k,k+1}\big)/2$; and the parameters $\mathbf{s}_k^2=\sigma_k^2+\sigma_{k+1}^2$ can be estimated from (13.13) and from the increments of the observable capital distribution curve of (13.14), namely $\widehat{\mathbf{s}}_k^2=2\widehat{\lambda}_{k,k+1}\big(\mathfrak{m}(k)-\mathfrak{m}(k+1)\big)$. For the decade 1990-99, these estimates are presented in Figure 13.6.

Finally, we make the following selections for estimating the variances:

$$\widehat{\sigma}_k^2=\frac{1}{4}\left(\widehat{\mathbf{s}}_{k-1}^2+\widehat{\mathbf{s}}_k^2\right),\quad k=2,\cdots,n-1\,,\qquad\text{and}\qquad \widehat{\sigma}_1^2=\frac{1}{2}\widehat{\mathbf{s}}_1^2\,,\quad \widehat{\sigma}_n^2=\frac{1}{2}\widehat{\mathbf{s}}_{n-1}^2\,.$$

□

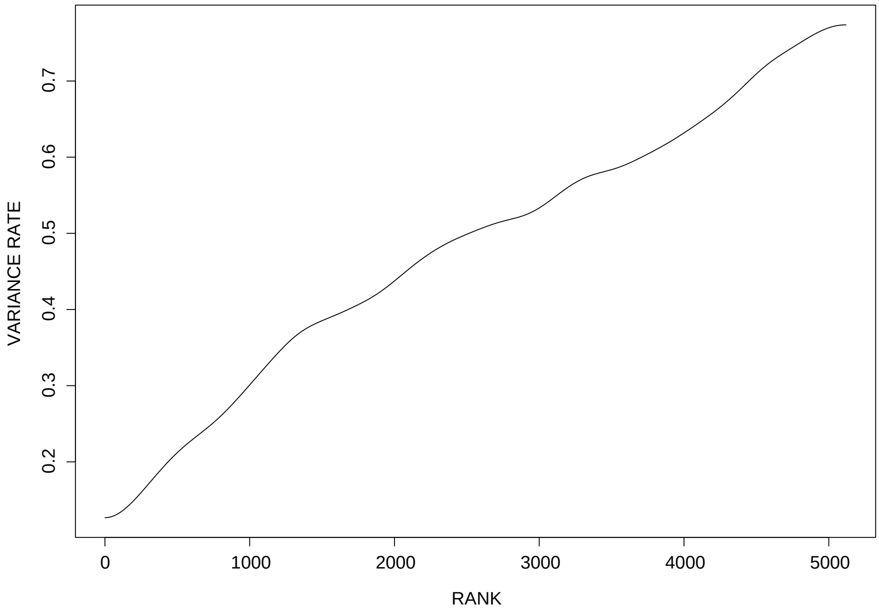

Figure 13.6: Smoothed annualized values of $\widehat{\mathbf{s}}_k^2$ for $k = 1, \ldots, 5119$. Calculated from 1990–1999 data.

13.4 Stability of the Capital Distribution

Let us now go back to (13.11); it can be seen that this leads to the convergence of the ranked market weights

$$\lim_{t \to \infty} \big(\mu_{(1)}(t), \ldots, \mu_{(n)}(t)\big) = (M_1, \ldots, M_n), \quad \text{in distribution} \tag{13.16}$$

to the random variables

$$M_n := \big(1 + e^{\xi_{n-1}} + \cdots + e^{\xi_1 + \cdots + \xi_{n-1}}\big)^{-1}, \quad \text{and} \quad M_k := M_n \, e^{\xi_k + \cdots + \xi_{n-1}} \tag{13.17}$$

for $k = 1, \ldots, n-1$. These are the long-term (steady-state) relative weights of the various stocks in the market, ranked from largest, M_1, to smallest, M_n. Again, we have from (13.16) the strong law of large numbers

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f\big(\mu_{(1)}(t), \ldots, \mu_{(n)}(t)\big) dt \;=\; \mathbb{E}\big(f(M_1, \ldots, M_n)\big), \quad \text{a.s.} \tag{13.18}$$

for every bounded and measurable $f : \Delta_+^n \to \mathbb{R}$. Note that (13.13) is a special case of this result, and that the function $\mathfrak{m}(\cdot)$ of (13.14) takes the form

$$\mathfrak{m}(k) \;=\; \mathbb{E}\big(\log(M_k)\big) \;=\; \sum_{\ell=k}^{n-1} \frac{1}{r_\ell} \;-\; \mathbb{E}\big(\log(1 + e^{\xi_{n-1}} + \cdots + e^{\xi_1 + \cdots + \xi_{n-1}})\big). \tag{13.19}$$

This is the good news; the bad news is that we do not know, in general, the joint distribution of the exponential random variables $\xi_1, \cdots, \xi_{n-1}$ in (13.11), so we cannot find that of $M_1, \cdots, M_n$ either. In particular, we cannot pin down the steady-state capital distribution function of (13.19), though we *do* know precisely its increments $\mathfrak{m}(k+1) - \mathfrak{m}(k) = -(1/r_k)$ and thus are able to estimate the slope of the steady-state capital distribution curve, as indeed we did in (13.15). In [BFK] (2005) a simple, certainty-equivalent approximation of the steady-state ranked market weights of (13.17) is carried out, and is used to study in detail the behavior of simple portfolios in such a model.

Remark 13.4. Open Question: What can be said about the joint distribution of the long-term (steady-state) relative market weights of (13.17)? Can it be characterized, computed, or approximated in a good way? What can be said about the fluctuations of the random variables $\log(M_k)$ with respect to their means $\mathfrak{m}(k)$ in (13.19)?

For answers to some of these questions for equal variances and large numbers of assets (in the limit as $n \to \infty$), see the important recent work of Pal & Pitman (2007) and Chatterjee & Pal (2007). $\square$

Remark 13.5. Research Question and Conjecture: Study the steady-state capital distribution curve of the volatility-stabilized model in (12.1). With $\alpha > 0$, check the validity of the following conjecture: the slope

$$\mathfrak{q}(k) \approx \frac{\mathfrak{m}(k) - \mathfrak{m}(k+1)}{\log k - \log(k+1)}$$

of the capital distribution $\mathfrak{m}(\cdot)$ at $\log k$, should be given as

$$\mathfrak{q}(k) \approx -4\gamma k \mathfrak{h}_k , \qquad \mathfrak{h}_k := \mathbb{E}\left(\frac{\log Q_{(k)} - \log Q_{(k+1)}}{Q_{(1)} + \cdots + Q_{(n)}} \right),$$

where $Q_{(1)} \geq \cdots \geq Q_{(n)}$ are the order statistics of a random sample from the chi-square distribution with $\kappa = 2(1+\alpha)$ degrees of freedom.

If this conjecture is correct, does $k\mathfrak{h}_k$ increase with k? $\square$

14 Some Concluding Remarks

We have surveyed a framework, called *Stochastic Portfolio Theory*, for studying the behavior of portfolio rules and for modeling and analyzing equity market structure. We have also exhibited simple conditions, such as "diversity" and "availability of intrinsic volatility", which can lead to arbitrages relative to the market.

These conditions are *descriptive* in nature, and can be tested from the predictable characteristics of the model posited for the market. In contrast, familiar assumptions, such as the existence of an equivalent martingale measure (EMM), are *normative* in nature; they *cannot* be decided on the basis of predictable characteristics in the model. In this vein, the Example 4.7, pp. 469-470 of [KK] (2007) is quite instructive.

The existence of such relative arbitrage is not the end of the world. Under reasonably general conditions, one can still work with appropriate "deflators" for the purposes of hedging contingent claims and of portfolio optimization, as we have tried to illustrate in Section 10.

Considerable computational tractability is lost, as the marvelous tool that is the EMM goes out the window. Nevertheless, big swaths of the field of Mathematical Finance remain totally or mostly intact; and completely new areas and issues, such as those of the "Abstract Markets" in Part IV of this survey, thrust themselves onto the scene.

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