Back-Running: Seeking and Hiding Fundamental Information in Order Flows*

Liyan Yang† Haoxiang Zhu‡

January 2016

Abstract

We model the strategic interaction between fundamental informed trading and order-flow informed trading. Adding to a two-period Kyle (1985) model, a "back-runner" observes a signal of the fundamental informed investor's period-1 order *after* the order is filled. Learning from past order-flow information, the back-runner competes with the fundamental investor in period 2. If order-flow information is accurate, the fundamental investor hides her information by randomizing her period-1 trade, resulting in a mixed-strategy equilibrium. A pure-strategy equilibrium obtains if order-flow information is inaccurate. Back-running delays price discovery and reduces fundamental information acquisition. Recent evidence on high-frequency trading supports our theoretical predictions.

JEL code: G14, G18
Keywords: order flow, order anticipation, high-frequency trading, price discovery, market liquidity

*For helpful discussions and comments, we thank Bradyn Breon-Drish, Mike Burkart, Henry Cao, David Easley, Itay Goldstein, Jungsuk Han, Eric Hughson, Leonid Kogan, Pete Kyle, Doron Levit, Wei Li, Albert Menkveld, Maureen O'Hara, Christine Parlour, Uday Rajan, Ioanid Rosu, Francesco Sangiorgi, Duane Seppi, Matthew Spiegel, Gunter Strobl, Yajun Wang, Mao Ye, Jerry Yu, and Bart Yueshen, as well as seminar and conference participants at Finance Theory Group Summer School, Northern Finance Association annual meeting, the 11th Central Bank Workshop on the Microstructure of Financial Markets, the 26th Annual Conference on Financial Economics and Accounting, 2016 ASSA annual meetings, Gothenburg University, Lund University, Stockholm School of Economics, and Wisconsin Business School. We especially thank Hyungjune Kang for providing excellent research assistance. The most updated version of this paper is posted at http://ssrn.com/abstract=2583915.

†Rotman School of Management, University of Toronto; liyan.yang@rotman.utoronto.ca.
‡MIT Sloan School of Management; zhuh@mit.edu.

Back-Running: Seeking and Hiding Fundamental Information in Order Flows

Abstract

We model the strategic interaction between fundamental informed trading and order-flow informed trading. Adding to a two-period Kyle (1985) model, a "back-runner" observes a signal of the fundamental informed investor's period-1 order *after* the order is filled. Learning from past order-flow information, the back-runner competes with the fundamental investor in period 2. If order-flow information is accurate, the fundamental investor hides her information by randomizing her period-1 trade, resulting in a mixed-strategy equilibrium. A pure-strategy equilibrium obtains if order-flow information is inaccurate. Back-running delays price discovery and reduces fundamental information acquisition. Recent evidence on high-frequency trading supports our theoretical predictions.

1 Introduction

This paper studies the strategic interaction between fundamental informed trading and order-flow informed trading, as well as its implications for market equilibrium outcomes. By order-flow informed trading, we refer to strategies that begin with no innate trading motives—be it fundamental information or liquidity needs—but instead first learn about other investors' order flows and then act accordingly.

A primary example of order-flow informed trading is "order anticipation" strategies. According to the Securities and Exchange Commission (2010, p. 54–55), order anticipation "involves any means to ascertain the existence of a large buyer (seller) that does not involve violation of a duty, misappropriation of information, or other misconduct. Examples include the employment of sophisticated pattern recognition software to ascertain from *publicly available information* the existence of a large buyer (seller), or the sophisticated use of orders to 'ping' different market centers in an attempt to locate and trade in front of large buyers and sellers [emphasis added]."

Always been controversial,[1] order anticipation strategies have recently attracted intense attention and generated heated debates in the context of high-frequency trading (HFT). In a colorful account of today's U.S. equity market, Lewis (2014) argues that high-frequency traders observe part of investors' orders on one exchange and "front-run" the remaining orders before they reach other exchanges.[2] Although most (reluctantly) agree that such strategies are legal in today's regulatory framework, many investors and regulators have expressed severe concerns that they could harm market quality and long-term investors.[3] For example, in its influential Concept Release on Equity Market Structure, Securities and Exchange Commission (2010, p. 56) asks: "Do commenters believe that order anticipation significantly detracts from market quality and harms institutional investors?"

To address important policy questions like this, we need to first address the more fun-

[1] For example, Harris (2003) writes "Order anticipators are *parasitic traders*. They profit only when they can prey on other traders [emphasis in original]."

[2] In its original sense, front-running refers to the illegal practice that a broker executes orders on his own account before executing a customer order. In recent discussions of market structure, this term is often used more broadly to refer to any type of trading strategy that takes advantage of order-flow information, including some academic papers that we will discuss shortly. When discussing these papers we use "front-running" to denote the broader meaning, as the original authors do.

[3] It should be noted that order anticipation strategies are not restricted to HFT; they also apply to other market participants such as broker-dealers who have such a technology. "The successful implementation of this strategy (order anticipation) depends less on low-latency communications than on high-quality pattern-recognition algorithms," remarks Harris (2013). "The order anticipation problem is thus not really an HFT problem."

damental questions of market equilibrium. For example, how do order anticipators take advantage of their superior order-flow information of fundamental investors (such as mutual funds and hedge funds)? How do these fundamental investors, in turn, respond to potential information leakage? How does the strategic interaction between these two types of traders affect market equilibrium and associated market quality?

In this paper, we take up this task. Our analysis builds on a simple theoretical model of strategic trading. We start from a standard two-period Kyle (1985) model, which has a "fundamental investor" who is informed of the true asset value, noise traders, and a competitive market maker. The novel part of our model is that we add a "back-runner" who begins with no fundamental information nor liquidity needs, but who receives a signal of the fundamental investor's order flow after that order is executed by the market maker. In Section 2 we provide some examples of how order-flow signals may be extracted.

The trading mechanism of this market is the same as the standard Kyle (1985) model. In the first period, only the fundamental investor and noise traders submit market orders, which are filled by the market maker at the market-clearing price. Only after the period-1 market clears does the back-runner observe the fundamental investor's period-1 order flow (with noise). In the second period, the fundamental investor, the back-runner, and noise traders all submit market orders, which are filled at a new market-clearing price.

We emphasize that it is the fundamental investor's order flow, not her information,[4] that is partially observed by the back-runner (otherwise, the back-runner would simply be another fundamental investor); and this order-flow information is observed ex post, not ex ante. This important feature is directly motivated by the "publicly available information" part in the SEC's definition above. For this reason, we believe "back-running" is a more realistic description of order-anticipation strategies than "front-running" because "back-running" acknowledges that order-flow information is not endowed instantly, but learned over time. As we discuss in Section 5, recent evidence supports the back-running interpretation.

The risk that order flow leaks valuable information substantially changes the fundamental investor's behavior. In particular, pure strategy equilibrium may no longer exist. To see why, note that in the extreme case that the back-runner perfectly observes the fundamental investor's past order flow, a pure strategy by the fundamental investor completely reveals her information to the back-runner. Clearly, creating a competitor in the next period harms the fundamental investor. As long as the back-runner's order-flow signal is sufficiently pre-

[4]Throughout this paper, we will use "her"/"she" to refer to the fundamental investor and use "his"/"he" to refer to the back-runner.

cise, playing a pure strategy is suboptimal for the fundamental investors. We show that in those situations the unique linear equilibrium is a mixed strategy equilibrium in which the fundamental investor adds an endogenous, normally-distributed noise order into her period-1 order flow to hide her information. This garbled order flow, in turn, makes it harder for the back-runner to infer the asset fundamental value. In other words, if investors face a high risk of information leakage, randomization is the best defense. This result echoes nicely Stiglitz (2014, p. 8)'s remark on high-frequency trading: "[T]he informed, knowing that there are those who are trying to extract information from observing (directly or indirectly) their actions, will go to great lengths to make it difficult for others to extract such information."

In contrast, if the back-runner's order-flow information is sufficiently noisy, he has a hard time inferring the fundamental investor's information anyway. In this case, the fundamental investor does not need to inject additional noise; she simply plays a pure strategy.

Our analysis points out a new channel—i.e., the amount of noise in the back-runner's signal—that determines whether a mixed strategy equilibrium or a pure strategy one should prevail in a Kyle-type auction game. In particular, if there is less *exogenous* noise in the back-runner's signal, the fundamental investor *endogenously* injects more noise into her own period-1 order flow. As a result, as the amount of noise in the back-runner's signal increases from 0 to ∞, the unique linear equilibrium switches from a mixed strategy equilibrium to a pure strategy one. Characterizing the endogenous switch between a mixed strategy equilibrium and a pure strategy one is the first, theoretical contribution of our paper.

The second, applied contribution of our paper is to investigate the implications of back-running for market quality and traders' welfare. The natural benchmark is a standard two-period Kyle model without the back-runner. Our results reveal that the presence of back-running delays price discovery. In the presence of back-running, the fundamental investor trades less aggressively and possibly adds noise in the first period, harming price discovery. Price discovery is improved in the second period, however, since the back-runner also has value-relevant information and trades with the fundamental investor.

Market liquidity, measured by the inverse of Kyle's lambda, is mixed. The first-period liquidity is generally improved because the more cautious trading of the fundamental investor weakens the market maker's adverse selection problem. But the presence of back-running can either improve or harm the second-period market liquidity. It will harm liquidity if the back-runner's order-flow signal is sufficiently precise, which means that his trading will inject more private information into the second-period order flow, aggravating the market maker's adverse selection problem.

Unsurprisingly, taking the two periods together, the fundamental investor suffers from the presence of back-running, but noise traders benefit from it. Because institutional investors like mutual funds, pension funds, and ETFs employ a wide variety of investment strategies, they may act as either fundamental investors or liquidity (noise) traders, depending on the context. Since the back-runner makes a positive expected profit, the net result is that the other two trader types suffer collectively. We thus confirm the suspicion by regulators that order-flow informed trading tends to harm institutional investors on average.

We consider endogenous information acquisition in an extension of the baseline model. The fundamental investor chooses to acquire a certain amount of fundamental information, and the back-runner simultaneously chooses the precision of order-flow information that he acquires. Our prior results are robust to endogenous information acquisition. We also find that a lower cost of acquiring order-flow information reduces the fundamental investor's incentive to acquire fundamental information.

The theoretical results on the behaviors of various market participants are supported by recent studies that link the activities of (certain) high-frequency traders (HFTs) to the execution performance of institutional investors. Relevant studies include van Kervel and Menkveld (2015), Tong (2015), and Korajczyk and Murphy (2014). Our theoretical prediction that back-running delays price discovery is directly supported by Weller (2015), and consistent with Brogaard, Hendershott, and Riordan (2014) and Hirschey (2013). These studies are discussed in detail in Section 5.

A practical implication of our results is that randomized execution strategies help institutional investors reduce information leakage. A simple way to implement randomization is to overlay mean-zero random perturbations to standard execution schedules such as time-weighted average price (TWAP) or volume-weighted average price (VWAP). Moreover, the appearance of market manipulation—selling and then buying at lower prices, or vice versa—could be part of an optimal execution strategy that aims to limit information leakage.

1.1 Relation to the literature

Our paper contributes to three branches of literature: theories on high-frequency trading, mixed strategies in trading models, and order-flow informed trading.

High-frequency trading. The recent theoretical literature on HFT typically assumes that high-frequency traders have information advantage. Relevant papers include Biais, Foucault, and Moinas (2015), Foucault, Hombert, and Rosu (2015), Hoffmann (2014), Budish, Cramton, and Shim (2015), and Jovanovic and Menkveld (2012). In those models, a high-frequency trader plays the dual role of being fast and being informed. In our model, the back-runner is not as informed as the fundamental investor, but the back-runner can collect information from the fundamental investor's trading behavior. It is the separation between fundamental information and order-flow information that gives rise to the interesting interactions and implications observed in our model. Another connection is that we endogenize the source of HFT's private information—through parsing public order flows—that is commonly assumed in existing HFT studies.

Mixed strategies in trading models. At a technical level, the model of our paper is closest to that of Huddart, Hughes, and Levine (2001), also an extension of Kyle (1985). Motivated by the mandatory disclosure of trades by firm insiders, they assume that the insider's orders are disclosed *publicly and perfectly* after being filled. They show that the only equilibrium in their setting is a mixed strategy one, for otherwise the market maker would perfectly infer the asset value, preventing any further trading profits of the insider. In their model the mandatory public disclosure unambiguously improves price discovery and market liquidity in each period.

Buffa (2013) studies disclosure of insider trades when the inside is risk-averse. His equilibrium with disclosure also features mixed strategies. In contrast to Huddart, Hughes, and Levine (2001), however, he shows that disclosing insider trades can harm price discovery by making the risk-averse insider trade less aggressively.

Our results differ from those of Huddart, Hughes, and Levine (2001) and Buffa (2013) in at least two important aspects. First, we identify the endogenous switching between the mixed strategy equilibrium and the pure strategy one, depending on the precision of the order-flow information. To the best of our knowledge, ours is the first model that presents a pure-mix strategy equilibrium switch among many extensions of Kyle (1985).[5] Second, while their models apply to public disclosure of insider trades, our model is much more suitable to analyze the *private* learning of order-flow information by proprietary firms such as HFTs. Some may view this model difference as small and inconsequential, but bringing the model a little closer to reality can substantially improve the applicability of the model. As we

[5] A few other studies identify mixed strategy equilibria in different settings. In a continuous-time extension of Glosten and Milgrom (1985) model, Back and Baruch (2004) show that there is a mixed strategy equilibrium in which the informed trader's strategy is a point process with stochastic intensity. Baruch and Glosten (2013) show that "flickering quotes" and "fleeting orders" can arise from a mixed strategy equilibrium in which quote providers repeatedly undercut each other. Yueshen (2015) shows that if market makers are not perfectly competitive and the number of market makers is uncertain, then market makers who are present use a mixed pricing strategy. These papers do not explore the question of trading on order-flow information or a switch between pure and mixed strategy equilibria.

have shown, private learning of order-flow information delays price discovery, opposite to the prediction of Huddart, Hughes, and Levine (2001). As discussed in Section 5, recent evidence from Weller (2015) supports our prediction.

Order-flow information. Among papers studying order-flow information, the one closest to ours is Madrigal (1996), who also considers a two-period Kyle (1985) model with an insider and a "(non-fundamental) speculator." The speculator observes part of the period-1 noise trading. Although Madrigal's model and ours are similar, he only considers pure strategy equilibria and does not verify the second-order condition. In fact, the second-order condition for his pure strategy equilibrium turns out to be violated when the speculator observes a precise signal of noise trading (hence infers the insider's trade accurately). Consequently, his result misses the mixed strategy equilibrium entirely, and hence misses how fundamental investors counteract information leakage by adding noise to order flows.

The mixed strategy equilibrium also matters a great deal for market quality implications. We show that when the back-runner's information of past order flows is accurate, only the mixed strategy equilibrium exists, and price discovery in the first period becomes *worse* than the standard Kyle model. For these parameter values, if one were to apply Madrigal's pure strategy equilibrium, one would conclude, incorrectly, that the presence of the (non-fundamental) speculator would improve price discovery in the first period, relative to the standard Kyle model. As we discuss in Section 5, recent evidence from Weller (2015) supports the prediction from our mixed strategy equilibrium.

Li (2014) models high-frequency trading "front-running," whereby multiple HFTs with various speeds observe the aggregate order flow *ex ante* with noise and front-run it before it reaches the market maker. In his model the informed trader has one trading opportunity and does not counter information leakage by adding noise.

Other earlier models exploring information about liquidity-driven order flows include Cao, Evans, and Lyons (2006), Bernhardt and Taub (2008), Attari, Mello, and Ruckes (2005), Brunnermeier and Pedersen (2005), and Carlin, Lobo, and Viswanathan (2007). Our model differs from them in two ways: (i) the relevant information is about asset fundamentals, not liquidity needs; and (ii) order-flow information is learned over time, not endowed instantly. As elaborated in Section 5, these differences have distinct empirical predictions. Evidence on HFT behaviors by van Kervel and Menkveld (2015) supports the premise and results of our back-running model. Moreover, the fundamental investor in our model optimally injects noise into her orders as camouflage, a feature absent in other studies in this category. Our price-discovery implication is supported by recent evidence from Weller (2015).

2 A Model of Back-Running

This section provides a model of back-running, based on the standard Kyle (1985) model. For ease of reference, main model variables are tabulated and explained in Appendix A. All proofs are in Appendix B.

2.1 Setup

There are two trading periods, $t = 1$ and $t = 2$. The timeline of the economy is described by Figure 1. A risky asset pays a liquidation value $v \sim N(p_0, \Sigma_0)$ at the end of period 2, where $p_0 \in \mathbb{R}$ and $\Sigma_0 > 0$. A single "fundamental investor" learns v at the start of the first period and places market orders x_1 and x_2 at the start of periods 1 and 2, respectively. Noise traders' net demands in the two periods are u_1 and u_2, both distributed $N(0, \sigma_u^2)$, with $\sigma_u > 0$. Random variables v, u_1 and u_2 are mutually independent. Asset prices p_1 and p_2 are set by a competitive market maker who observes the total order flow at each period, y_1 and y_2, and sets the price equal to the posterior expectation of v given public information.

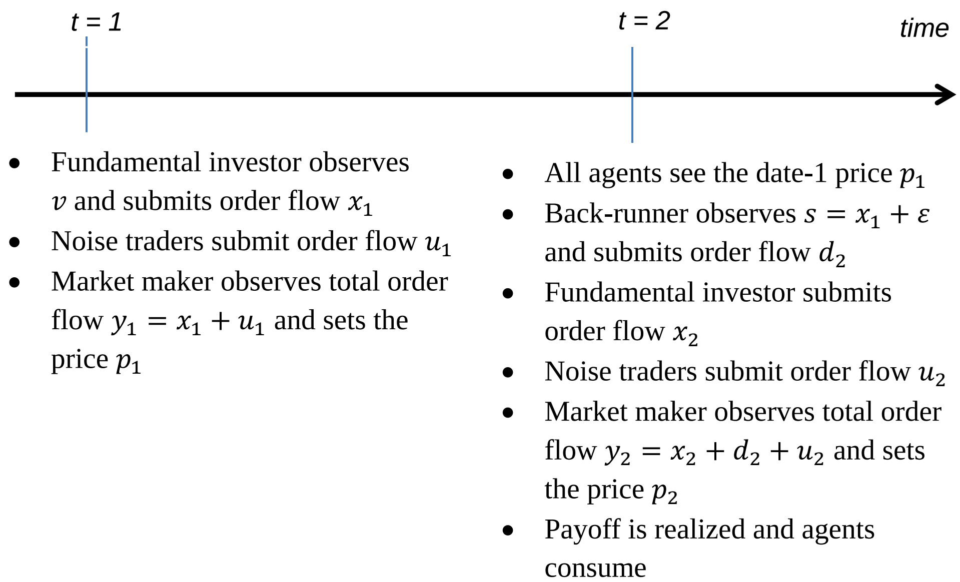
The main difference from a standard Kyle model is that there is a "back-runner " who can extract private information from public order flows and trades on this private information in period 2. We call this trader a back-runner instead of a "front-runner" to highlight that his information is learned over time, not endowed instantly. Specifically, after seeing the aggregate period-1 order flow y_1, which is public information in period 2, the back-runner observes a signal about the fundamental investor's period-1 trades x_1 as follows:

$$s = x_1 + \varepsilon, \tag{1}$$

where $\varepsilon \sim N(0, \sigma_\varepsilon^2)$, where $\sigma_\varepsilon \in [0, \infty]$, is independent of all other random variables (v, u_1 and u_2). Parameter σ_ε controls the information quality of the signal s—a larger σ_ε means less accurate information about x_1. In particular, we deliberately allow σ_ε to take values of 0 and ∞, which respectively corresponds to the case in which s perfectly reveals x_1 and the case in which s reveals nothing about x_1.

In practice, a back-runner has a number of ways to obtain the signal s, and here are two stylized examples. First, execution algorithms used by institutional investors may leave "footprints" that are subsequently detected by more sophisticated algorithms. As an extremely simple example, consider a "time-weighted average price" (TWAP) algorithm that splits a large order of 50,000 shares into 500 small orders of 100 shares each and submits one small order every second. Such an execution strategy is detectable by another algorithm due to the regularity of the timing and quantity of the series of orders (also see Easley,

Figure 1: Model Timeline



de Prado, and O'Hara (2012) for a discussion of this point). The second example is that the back-runner could take advantage of the behavior biases of individual investors to collect order-flow information about noise trading u_1, which can be translated into a signal of x_1 given $y_1 = x_1 + u_1$. Bhattacharya, Holden, and Jacobsen (2012) find evidence that "stock traders focus on round numbers as cognitive reference points for value." To the extent that individual investors are more likely than computer algorithms to anchor on round numbers, the clustering of trading volume or limit orders near round numbers could be a signal of the degree of uninformed noise trading. Of course, in reality the algorithms used by back-runners are much more complicated and less understood than those discussed here, but the intuition is similar.

After receiving the signal s, the back-runner places a market order d_2 in period 2. As a result, the market maker receives an aggregate order flow

$$y_2 = x_2 + d_2 + u_2. \tag{2}$$

Of course, in period 1, the aggregate order flow is

$$y_1 = x_1 + u_1, \tag{3}$$

since during period 1 the back-runner has no private information and does not send any order. The weak-form-efficiency pricing rule of the market maker implies

$$p_1 = E\left(v|y_1\right) \text{ and } p_2 = E\left(v|y_1, y_2\right). \tag{4}$$

At the end of period 2, all agents receive their payoffs and consume, and the economy ends.

As discussed in the introduction, a practical interpretation of this back-runner is that he uses advanced technology and processes public information better than the general public, represented by the market maker in the model. Such superior ability of processing public information has long been recognized in the literature. For example, Kim and Verrecchia (1994) argue that savvy market participants, such as asset managers and analysts, can process information better than the market by converting a firm's noisy public signals (e.g., earnings announcements) into more accurate information. Engelberg, Reed, and Ringgenberg (2012) show that a significant portion of short sellers profitability actually comes from their skills in analyzing public information. Our objective is to explore how the presence of the back-runner—a trader who has superior skills in processing public trading data to extract the patterns of trades—affects the trading strategies of the fundamental investor (such as pension funds, mutual funds, or hedge funds) as well as the resulting market equilibrium outcomes.

2.2 Equilibrium Definitions

A *perfect Bayesian equilibrium* of the trading game is given by a strategy profile

$$\{x_1^*(v), x_2^*(v, p_1, x_1), d_2^*(s, p_1), p_1^*(y_1), p_2^*(y_1, y_2)\},$$

that satisfies:

1. Profit maximization:

$$\begin{aligned} & x_2^* \in \arg\max_{x_2} E\left[x_2(v-p_2)\,|v, p_1, x_1\right], \\ & d_2^* \in \arg\max_{d_2} E\left[d_2(v-p_2)\,|s, p_1\right], \\ \textit{and} \quad & x_1^* \in \arg\max_{x_1} E\left[x_1(v-p_1) + x_2^*(v-p_2)\,|v\right]. \end{aligned}$$

2. Market efficiency: p_1 and p_2 are determined according to equation (4).

Note that in the perfect Bayesian equilibrium, we allow mixed strategies, that is, in principle the strategies x_1^*, x_2^*, and d_2^* could be probability distributions over quantities. It turns out that in the equilibrium we characterize shortly, x_1 can involve mixing, but x_2 and d_2 are both pure strategies.

We will focus on linear equilibria, i.e., the trading strategies and pricing functions are linear. Formally, a *linear equilibrium* is defined as a perfect Bayesian equilibrium in which there exist constants

$$(\beta_{v,1}, \beta_{v,2}, \beta_{x_1}, \beta_{y_1}, \delta_s, \delta_{y_1}, \lambda_1, \lambda_2) \in \mathbb{R}^8 \quad \textit{and} \quad \sigma_z \geq 0,$$

such that

$$x_1 = \beta_{v,1}(v - p_0) + z \text{ with } z \sim N\left(0, \sigma_z^2\right), \tag{5}$$

$$x_2 = \beta_{v,2}(v - p_1) - \beta_{x_1} x_1 + \beta_{y_1} y_1, \tag{6}$$

$$d_2 = \delta_s s - \delta_{y_1} y_1, \tag{7}$$

$$p_1 = p_0 + \lambda_1 y_1 \text{ with } y_1 = x_1 + u_1, \tag{8}$$

$$p_2 = p_1 + \lambda_2 y_2 \text{ with } y_2 = x_2 + d_2 + u_2, \tag{9}$$

where z is independent of all other random variables $(v, u_1, u_2, \varepsilon)$.

Equations (5)–(9) are intuitive. Equations (5)–(7) simply say that the fundamental investor and the back-runner trade on their information advantage. Importantly, our specification (5) allows the fundamental investor to play a mixed strategy in period 1. We have followed Huddart, Hughes, and Levine (2001) and restricted attention to normally distributed z in order to maintain tractability. If $\sigma_z = 0$, the fundamental investor plays a pure strategy in period 1, and we refer to the resulting linear equilibrium as a *pure strategy equilibrium.* If $\sigma_z > 0$, the fundamental investor plays a mixed strategy in period 1, and we refer to the resulting linear equilibrium as a *mixed strategy equilibrium.* As we show shortly, by adding noise into her orders, the fundamental investor limits the back-runner's ability to infer x_1 and hence v. To an outside observer, the endogenously added noise z may look like exogenous noise trading.

Although in principle the fundamental investor and the back-runner can play mixed strategies in period 2, we show later that using mixed strategies in period 2 is suboptimal in equilibrium. Thus, the linear period-2 trading strategies specified in equations (6) and (7) are without loss of generality. They are also the most general linear form, as each equation spans the information set of the relevant trader in the relevant period. Note that at this stage we do not require that β_{x_1}, β_{y_1}, δ_s or δ_{y_1} be positive, although in equilibrium they will be positive. (One can also show that the back-runner does not wish to play a mixed strategy in period 1.)

Equation (6) has three terms. The first term $\beta_{v,2}(v - p_1)$ captures how aggressively the fundamental investor trades on her information advantage about v. The other two terms $-\beta_{x_1} x_1$ and $\beta_{y_1} y_1$ say that the fundamental investor potentially adjusts her period-2 market order by using lagged information x_1 and y_1. Because the back-runner generally uses y_1 and his signal s about x_1 to form his period-2 order (see equation (7)), the fundamental investor takes advantage of this predictive pattern by using x_1 and y_1 in her period-2 order as well.

In equilibrium characterized later, the conjectured strategy in equation (7) can also be written alternatively as:

$$d_2 = \alpha \left[E(v|s, y_1) - E(v|y_1) \right], \tag{10}$$

for some constant $\alpha > 0$ (see Appendix B.1 for a proof). That is, the back-runner's order is proportional to his information advantage relative to the market maker's. By the joint normality of s and y_1, this alternative form implies that d_2 is linear in s and y_1. We nonetheless start with (7) because it is the most general and does not impose any structure as (10) does. We start with equation (6) for a similar reason.

The pricing equations (8) and (9) state that the price in each period is equal to the expected value of v before trading, adjusted by the information carried by the new order flow. Although the conjectured p_2 may in principle depend on y_1, in equilibrium p_1 already incorporates all information of y_1.[6] Thus, we can start with (9).

2.3 Equilibrium Derivation

We now derive by backward induction all possible linear equilibria. Along the derivations, we will see that the distinction between pure strategy and mixed strategy equilibria lies only in the conditions characterizing the fundamental investor's period-1 decision. Explicit statements of the equilibria and their properties are presented in the next subsection.

Fundamental investor's date-2 problem. In period 2, the fundamental investor has information $\{v, p_1, x_1\}$. Given $\lambda_1 \neq 0$, which holds in equilibrium, the fundamental investor can infer y_1 from p_1 by equation (8). Using equations (7) and (9), we can compute

$$E\left[x_2 (v - p_2) | v, p_1, x_1\right] = -\lambda_2 x_2^2 + \left[v - p_1 - \lambda_2 \left(\delta_s x_1 - \delta_{y_1} y_1\right)\right] x_2. \tag{11}$$

Taking the first-order-condition (FOC) results in the solution as follows:

$$x_2 = \frac{v - p_1}{2\lambda_2} - \frac{\delta_s}{2} x_1 + \frac{\delta_{y_1}}{2} y_1. \tag{12}$$

The second-order-condition (SOC) is[7]

$$\lambda_2 > 0. \tag{13}$$

Equation (12) also implies that the fundamental investor optimally chooses to play a pure strategy in equilibrium, which verifies our conjectured pure strategy specification (6).

[6] Strictly speaking the most general form is $p_2 = p_1 + \lambda_2 \left[y_2 - E(y_2|y_1)\right]$. But in equilibrium we can show that $E(y_2|y_1) = 0$, so the more general form reduces to (9).

[7] The SOC cannot be $\lambda_2 = 0$, because otherwise, we have $p_2 = p_1 = p_0 + \lambda_1 y_1 = p_0 + \lambda_1 (x_1 + u_1)$, and thus $E(p_2|v) = p_0 + \lambda_1 x_1$, which means that the fundamental investor can choose x_1 and x_2 to make infinite profit in period 2. Thus, in any linear equilibrium, we must have $\lambda_2 > 0$.

Comparing equation (12) with the conjectured strategy (6), we have

$$\beta_{v,2} = \frac{1}{2\lambda_2}, \beta_{x_1} = \frac{\delta_s}{2} \text{ and } \beta_{y_1} = \frac{\delta_{y_1}}{2}. \tag{14}$$

Let $\pi_{F,2} = x_2 (v - p_2)$ denote the fundamental investor's profit that is directly attributable to her period-2 trade. Inserting (12) into (11) yields

$$E(\pi_{F,2}|v, p_1, x_1) = \frac{[v - p_1 - \lambda_2(\delta_s x_1 - \delta_{y_1} y_1)]^2}{4\lambda_2}. \tag{15}$$

Back-runner's date-2 problem. In period 2, the back-runner chooses d_2 to maximize $E(\pi_{B,2}|s, p_1)$, where

$$\pi_{B,2} = d_2 (v - p_2). \tag{16}$$

Using (6) and (9), we can compute the FOC, which delivers

$$d_2 = \frac{(1 - \lambda_2\beta_{v,2}) E(v - p_1|s, y_1) - \lambda_2\beta_{y_1} y_1 + \lambda_2\beta_{x_1} E(x_1|s, y_1)}{2\lambda_2}. \tag{17}$$

The SOC is still $\lambda_2 > 0$, as given by (13) in the fundamental investor's problem. Again, equation (17) means that the back-runner optimally chooses to play a pure strategy in a linear equilibrium.

We then employ the projection theorem and equations (1), (3), and (5) to find out the expressions of $E(v - p_1|s, y_1)$ and $E(x_1|s, y_1)$, which are in turn inserted into (17) to express d_2 as a linear function of s and y_1. Finally, we compare this expression with the conjectured strategy (7) to arrive at the following two equations:

$$\begin{aligned}
\delta_s &= \frac{\left[(1 - \lambda_2\beta_{v,2}) \frac{\beta_{v,1}\Sigma_0}{\beta_{v,1}^2\Sigma_0 + \sigma_z^2} + \lambda_2\beta_{x_1}\right] \frac{\sigma_\varepsilon^{-2}}{\left(\beta_{v,1}^2\Sigma_0 + \sigma_z^2\right)^{-1} + \sigma_\varepsilon^{-2} + \sigma_u^{-2}}}{2\lambda_2}, \\
\delta_{y_1} &= -\delta_s \frac{\sigma_u^{-2}}{\sigma_\varepsilon^{-2}} + \frac{\lambda_1 (1 - \lambda_2\beta_{v,2}) + \lambda_2\beta_{y_1}}{2\lambda_2}.
\end{aligned}$$

Using (14), we can further simplify the above two equations as follows:

$$\delta_s = \frac{\frac{\sigma_\varepsilon^{-2}}{\left(\beta_{v,1}^2\Sigma_0 + \sigma_z^2\right)^{-1} + \sigma_\varepsilon^{-2} + \sigma_u^{-2}}}{4 - \frac{\sigma_\varepsilon^{-2}}{\left(\beta_{v,1}^2\Sigma_0 + \sigma_z^2\right)^{-1} + \sigma_\varepsilon^{-2} + \sigma_u^{-2}}} \frac{\beta_{v,1}\Sigma_0}{\lambda_2\left(\beta_{v,1}^2\Sigma_0 + \sigma_z^2\right)}, \tag{18}$$

$$\delta_{y_1} = \frac{\lambda_1}{3\lambda_2} - \delta_s \frac{4\sigma_\varepsilon^2}{3\sigma_u^2}. \tag{19}$$

Market maker's decisions. In period 1, the market maker sees the aggregate order flow y_1 and sets $p_1 = E(v|y_1)$. Accordingly, we have $\lambda_1 = \frac{Cov(v,y_1)}{Var(y_1)}$. By equation (5) and the projection theorem, we can compute

$$\lambda_1 = \frac{Cov(v, y_1)}{Var(y_1)} = \frac{\beta_{v,1}\Sigma_0}{\beta_{v,1}^2\Sigma_0 + \sigma_z^2 + \sigma_u^2}. \tag{20}$$

Similarly, in period 2, the market maker sees $\{y_1, y_2\}$ and sets $p_2 = E(v|y_1, y_2)$. By equations (6), (7), and (14) and applying the projection theorem, we have

$$\begin{aligned}\lambda_2 &= \frac{Cov(v, y_2|y_1)}{Var(y_2|y_1)} \\ &= \frac{\left(\frac{1}{2\lambda_2} + \frac{\delta_s}{2}\beta_{v,1}\right)\Sigma_0 - \frac{\beta_{v,1}\Sigma_0\left[\left(\frac{1}{2\lambda_2}+\frac{\delta_s}{2}\beta_{v,1}\right)\beta_{v,1}\Sigma_0+\frac{\delta_s}{2}\sigma_z^2\right]}{\beta_{v,1}^2\Sigma_0+\sigma_z^2+\sigma_u^2}}{\left(\frac{1}{2\lambda_2} + \frac{\delta_s}{2}\beta_{v,1}\right)^2\Sigma_0 + \frac{\delta_s^2}{4}\sigma_z^2 + \delta_s\sigma_\varepsilon^2 + \sigma_u^2 - \frac{\left[\left(\frac{1}{2\lambda_2}+\frac{\delta_s}{2}\beta_{v,1}\right)\beta_{v,1}\Sigma_0+\frac{\delta_s}{2}\sigma_z^2\right]^2}{\beta_{v,1}^2\Sigma_0+\sigma_z^2+\sigma_u^2}}. \end{aligned} \tag{21}$$

Fundamental investor's date-1 problem. We denote by $\pi_{F,1} = x_1(v - p_1)$ the fundamental investor's profit that comes from her period-1 trade. In period 1, the fundamental investor chooses x_1 to maximize

$$E(\pi_{F,1} + \pi_{F,2}|v) = x_1 E(v - p_1|v) + E\left[\left.\frac{[v - p_1 - \lambda_2(\delta_s x_1 - \delta_{y_1} y_1)]^2}{4\lambda_2}\right| v\right],$$

where the equality follows from equation (15). Using (8), we can further express $E(\pi_{F,1} + \pi_{F,2}|v)$ as follows:

$$\begin{aligned} E(\pi_{F,1} + \pi_{F,2}|v) &= -\left[\lambda_1 - \frac{(\lambda_1 + \lambda_2\delta_s - \lambda_2\delta_{y_1})^2}{4\lambda_2}\right]x_1^2 \\ &\quad + \left[1 - \frac{\lambda_1 + \lambda_2\delta_s - \lambda_2\delta_{y_1}}{2\lambda_2}\right](v - p_0)x_1 \\ &\quad + \frac{(v - p_0)^2 + \sigma_u^2(\lambda_1 - \lambda_2\delta_{y_1})^2}{4\lambda_2}. \end{aligned} \tag{22}$$

Depending on whether the fundamental investor plays a mixed or a pure strategy (i.e., whether σ_z is equal to 0), we have two cases:

Case 1. Mixed Strategy ($\sigma_z > 0$)

For a mixed strategy to sustain in equilibrium, the fundamental investor has to be indifferent between any realized pure strategy. This in turn means that coefficients on x_1^2 and x_1 in (22) are equal to zero, that is,

$$\lambda_1 - \frac{(\lambda_1 + \lambda_2\delta_s - \lambda_2\delta_{y_1})^2}{4\lambda_2} = 0 \text{ and } 1 - \frac{\lambda_1 + \lambda_2\delta_s - \lambda_2\delta_{y_1}}{2\lambda_2} = 0.$$

These two equations, together with equation (19), imply

$$\lambda_1 = \lambda_2 \text{ and } \delta_s = \frac{\frac{4}{3}}{1 + \frac{4\sigma_\varepsilon^2}{3\sigma_u^2}}. \tag{23}$$

Case 2. Pure Strategy ($\sigma_z = 0$)

When the fundamental investor plays a pure strategy, $z = 0$ (and $\sigma_z = 0$) in the conjectured strategy, and thus (5) degenerates to $x_1 = \beta_{v,1}(v - p_0)$. The FOC of (22) yields

$$x_1 = \frac{\left(1 - \frac{\lambda_1+\lambda_2\delta_s-\lambda_2\delta_{y_1}}{2\lambda_2}\right)}{2\left[\lambda_1 - \frac{\left(\lambda_1+\lambda_2\delta_s-\lambda_2\delta_{y_1}\right)^2}{4\lambda_2}\right]}(v - p_0),$$

which, compared with the conjectured pure strategy $x_1 = \beta_{v,1}(v - p_0)$, implies

$$\beta_{v,1} = \frac{1 - \frac{\lambda_1+\lambda_2\delta_s-\lambda_2\delta_{y_1}}{2\lambda_2}}{2\left[\lambda_1 - \frac{\left(\lambda_1+\lambda_2\delta_s-\lambda_2\delta_{y_1}\right)^2}{4\lambda_2}\right]}. \tag{24}$$

The SOC is

$$\lambda_1 - \frac{\left(\lambda_1 + \lambda_2\delta_s - \lambda_2\delta_{y_1}\right)^2}{4\lambda_2} > 0. \tag{25}$$

2.4 Equilibrium Characterization and Properties

A mixed strategy equilibrium is characterized by equations (14), (18), (19), (20), (21), and (23), together with one SOC, $\lambda_2 > 0$ (given by (13)). These conditions jointly define a system that determine nine unknowns, $\sigma_z, \beta_{v,1}, \beta_{v,2}, \beta_{x_1}, \beta_{y_1}, \delta_s, \delta_{y_1}, \lambda_1$, and λ_2. The following proposition formally characterizes a linear mixed strategy equilibrium.

Proposition 1 (Mixed Strategy Equilibrium). *Let $\gamma \equiv \frac{\sigma_\varepsilon}{\sigma_u}$. If and only if $\gamma < \frac{\sqrt{\sqrt{17}-4}}{2} \approx 0.175$, there exists a linear mixed strategy equilibrium, and it is specified by equations (5)–(9), where*

$$\begin{aligned}
\sigma_z &= \sigma_u\sqrt{\frac{(1+4\gamma^2)(1-32\gamma^2-16\gamma^4)}{(3+4\gamma^2)(13+40\gamma^2+16\gamma^4)}}, \\
\beta_{v,1} &= \frac{\sigma_u}{\sqrt{\Sigma_0}}\sqrt{\frac{1-4\gamma^2-(3+4\gamma^2)\frac{\sigma_z^2}{\sigma_u^2}}{3+4\gamma^2}}, \\
\lambda_1 &= \lambda_2 = \frac{\beta_{v,1}\Sigma_0}{\beta_{v,1}^2\Sigma_0 + \sigma_z^2 + \sigma_u^2} > 0, \\
\beta_{v,2} &= \frac{1}{2\lambda_2}, \delta_s = \frac{4}{3+4\gamma^2}, \delta_{y_1} = \frac{1-4\gamma^2\delta_s}{3}, \\
\beta_{x_1} &= \frac{\delta_s}{2} \text{ and } \beta_{y_1} = \frac{\delta_{y_1}}{2}.
\end{aligned}$$

When it exists, this equilibrium is the unique linear mixed strategy equilibrium.

To illustrate the intuition of the equilibrium strategies, it is useful to explicitly decompose

d_2 as follows:

$$\begin{aligned} d_2 &= \delta_s(x_1+\varepsilon) - \delta_{y_1}(x_1+u_1) = (\delta_s - \delta_{y_1})x_1 + \delta_s\varepsilon - \delta_{y_1}u_1 \\ &= x_1 + \delta_s\varepsilon - \delta_{y_1}u_1 = \beta_{v,1}(v-p_0) + (\delta_s\varepsilon + z) - \delta_{y_1}u_1, \end{aligned} \tag{26}$$

where we have used the fact that $\delta_s - \delta_{y_1} = 1$ in equilibrium. Equation (26) says that the back-runner's order d_2 consists of three parts. The first part is the fundamental investor's order x_1 in period 1. The second part, $\delta_s\varepsilon + z$, reflects the imprecision of his signal, caused by both the exogenous noise ε in his signal-processing technology and the endogenous noise z added by the fundamental investor. The third part, $-\delta_{y_1}u_1$, says that the back-runner trades against the period-1 noise demand u_1, which is profitable in expectation because the back-runner can tell x_1 from u_1 better than the market maker does. Note that equation (26) should be read as purely as a decomposition but not the strategy used by the back-runner, as v, ε, z, and u_1 are not separately observable to him.

In the mixed strategy equilibrium, x_1, x_2, $v - p_0$, and $v - p_1$ need not always have the same sign. For example, if $v > p_0$ but z is sufficiently negative, the fundamental investor ends up selling in period 1 (with $x_1 < 0$), before purchasing in period 2 ($x_2 > 0$). While such a pattern in the data may raise red flags of potential "manipulation" (trading in the opposite direction of the true intention), it could simply be part of an optimal execution strategy that involves randomizing.

Proposition 1 reveals that a mixed strategy equilibrium exists if and only if the size σ_ε of the noise in the back-runner's signal is sufficiently small relative to σ_u. This result is natural and intuitive. A small σ_ε implies that the back-runner can observe x_1 relatively accurately. The back-runner will in turn compete aggressively with the fundamental investor in period 2, which reduces the fundamental investor's profit substantially. Worried about information leakage, the fundamental investor optimally plays a mixed strategy in period 1 by injecting an endogenous noise z into her order x_1, with σ_z uniquely determined in equilibrium. This garbled x_1 limits the back-runner's ability to learn about v. In other words, if the back-runner's order-parsing technology is accurate enough, randomization is the fundamental investor's best camouflage.

Conversely, if σ_ε is sufficiently large already, the fundamental investor retains much of her information advantage, and further obscuring x_1 is unnecessary. In this case a linear pure strategy equilibrium, characterized shortly, would be more natural.

Looked another way, all else equal, the mixed strategy equilibrium obtains if and only if σ_u is sufficiently *large*. Traditional Kyle-type models would not generate this result, as noise trading provides camouflage for the informed investor. In our model, however, a large

σ_u confuses only the market maker, not the back-runner. Thus, more noise trading implies a higher profit for the fundamental investor and hence a stronger incentive to retain her proprietary information by adding noise. A natural implication of this observation is that the exogenous noise σ_u reinforces the endogenous noise σ_z.

The threshold value for the existence of the mixed strategy equilibrium in our two-period model is $\sigma_\varepsilon/\sigma_u \approx 17.5\%$; whether it is large or small is an empirical question. In a model with more periods, one would expect this threshold to increase, because the fundamental investor has more time to trade on her information and hence has a stronger incentive to prevent information leakage. Such N-period extension turns out to be intractable due to the path dependence of the strategies. That said, we have solved a simpler two-period extension in which $Var\left(u_1\right) < Var\left(u_2\right)$, a specification that is previously used by Brunnermeier (2005). The larger noise-trading variance in the second period is meant to represent a longer trading interval (e.g., one hour versus one minute) or a larger market (multiple exchanges versus a single exchange). In this extension, not reported to conserve space, we find that mixed strategies indeed apply more often if $Var\left(u_2\right)/Var\left(u_1\right)$ is larger. For example, if $\sqrt{Var\left(u_2\right)/Var\left(u_1\right)} = 2$, the threshold for mixed strategy equilibrium satisfies $\sigma_\varepsilon/\sqrt{Var(u_1)} \approx 0.44$; and if $\sqrt{Var\left(u_2\right)/Var\left(u_1\right)} = 5$, the threshold for mixed strategy equilibrium satisfies $\sigma_\varepsilon/\sqrt{Var(u_1)} \approx 0.49$.

Now we turn to pure strategy equilibria. In a pure strategy equilibrium, we have $\sigma_z = 0$. This type of equilibrium is characterized by equations (14), (18), (19), (20), (21), and (24), together with two SOC's, (13) and (25). These conditions jointly define a system that determine eight unknowns, $\beta_{v,1}, \beta_{v,2}, \beta_{x_1}, \beta_{y_1}, \delta_s, \delta_{y_1}, \lambda_1$, and λ_2. The following proposition formally characterizes a linear pure strategy equilibrium.

Proposition 2 (Pure Strategy Equilibrium)**.** *A linear pure strategy equilibrium is characterized by equations (5)–(9) with $\sigma_z = 0$ as well as the following two conditions on $\beta_{v,1} \in \left(0, \frac{\sigma_u}{\sqrt{\Sigma_0}}\right]$:*
(1) $\beta_{v,1}^2$ solves the 7^{th} order polynomial:

$$f\left(\beta_{v,1}^2\right) = A_7\beta_{v,1}^{14} + A_6\beta_{v,1}^{12} + A_5\beta_{v,1}^{10} + A_4\beta_{v,1}^{8} + A_3\beta_{v,1}^{6} + A_2\beta_{v,1}^{4} + A_1\beta_{v,1}^{2} + A_0 = 0,$$

where the coefficients A's are given by equations (B14)–(B21) in Appendix B; and
(2) The following SOC (i.e., (25)) is satisfied:

$$\lambda_1 - \frac{\left(\lambda_1 + \lambda_2\delta_s - \lambda_2\delta_{y_1}\right)^2}{4\lambda_2} > 0,$$

where $\lambda_1, \lambda_2, \delta_s$, and δ_{y_1} are expressed as functions of $\beta_{v,1}$ as follows:

$$
\begin{aligned}
\lambda_1 &= \frac{\beta_{v,1}\Sigma_0}{\beta_{v,1}^2\Sigma_0 + \sigma_u^2}, \\
\lambda_2 &= \sqrt{\Sigma_0 \frac{\left(2\sigma_u^4 + 4\sigma_\varepsilon^4 + 5\sigma_u^2\sigma_\varepsilon^2\right)\Sigma_0^2\beta_{v,1}^4 + \left(8\sigma_u^2\sigma_\varepsilon^4 + 5\sigma_u^4\sigma_\varepsilon^2\right)\Sigma_0\beta_{v,1}^2 + 4\sigma_u^4\sigma_\varepsilon^4}{\left(\beta_{v,1}^2\Sigma_0 + \sigma_u^2\right)\left(3\sigma_u^2\Sigma_0\beta_{v,1}^2 + 4\sigma_\varepsilon^2\Sigma_0\beta_{v,1}^2 + 4\sigma_u^2\sigma_\varepsilon^2\right)^2}}, \\
\delta_s &= \frac{\beta_{v,1}\Sigma_0\sigma_u^2}{\lambda_2\left[\left(3\sigma_u^2 + 4\sigma_\varepsilon^2\right)\Sigma_0\beta_{v,1}^2 + 4\sigma_u^2\sigma_\varepsilon^2\right]}, \\
\delta_{y_1} &= \frac{\lambda_1}{3\lambda_2} - \frac{4\sigma_\varepsilon^2}{3\sigma_u^2}\delta_s.
\end{aligned}
$$

Propositions 1 and 2 respectively characterize mixed strategy and pure strategy equilibria. The following proposition provides sufficient conditions under which either equilibrium prevails as the unique one among linear equilibria.

Proposition 3 (Mixed vs. Pure Strategy Equilibria)**.** *If the back-runner has a sufficiently precise signal about x_1 (i.e., σ_ε^2 is sufficiently small), there is no pure strategy equilibrium, and the unique linear strategy equilibrium is the mixed strategy equilibrium characterized by Proposition 1. If the back-runner has a sufficiently noisy signal about x_1 (i.e., σ_ε^2 is sufficiently large), there is no mixed strategy equilibrium, and there is a unique pure strategy equilibrium characterized by Proposition 2.*

Given Proposition 1 and the discussion of its properties, the mixed strategy part of Proposition 3 is relatively straightforward. The existence of a pure strategy equilibrium for a sufficiently large σ_ε is also natural, as in this case the back-runner's signal has little information and does not deter the fundamental investor from using a pure strategy. In fact, as $\sigma_\varepsilon \uparrow \infty$ our setting degenerates to a standard two-period Kyle (1985) setting, and the unique linear equilibrium in our model indeed converges to the pure strategy equilibrium of Kyle (1985). This result is shown in the following corollary.

Corollary 1. *As $\sigma_\varepsilon \to \infty$, the linear equilibrium in the two-period economy with a back-runner converges to the linear equilibrium in the standard two-period Kyle model.*

Proposition 3 analytically proves the uniqueness of a linear equilibrium only for sufficiently small or sufficiently large values of σ_ε^2. It would be desirable to generalize this uniqueness result to any value of σ_ε, but we have not managed to do so due to the complexity of the 7^{th} order polynomial characterizing a pure strategy equilibrium in Proposition 2. In particular, given Proposition 1, a reasonable conjecture is that the boundary between pure and mix strategy equilibria is at $\frac{\sigma_\varepsilon}{\sigma_u} = \frac{\sqrt{\sqrt{17}-4}}{2}$. This conjecture, albeit not formally proven, seems to hold numerically. That is, if $\frac{\sigma_\varepsilon}{\sigma_u} < \frac{\sqrt{\sqrt{17}-4}}{2}$, only a mixed strategy linear equilibrium exists, and if $\frac{\sigma_\varepsilon}{\sigma_u} \geq \frac{\sqrt{\sqrt{17}-4}}{2}$, only a pure strategy linear equilibrium exists. Either way, the linear equilibrium seems unique for all parameter values.

Propositions 1 and 2 suggest the following three-step algorithm to compute all possible linear equilibria:

Step 1: Compute all the positive root of the polynomial $f\left(\beta_{v,1}^2\right) = 0$ in Proposition 2. Retain the values of $\beta_{v,1} \in \left(0, \frac{\sigma_u}{\sqrt{\Sigma_0}}\right]$ to serve as candidates for a pure strategy equilibrium.

Step 2: For each $\beta_{v,1}$ retained in Step 1, check whether the SOC in Proposition 2 is satisfied. If yes, then it is a pure strategy equilibrium; otherwise, it is not.

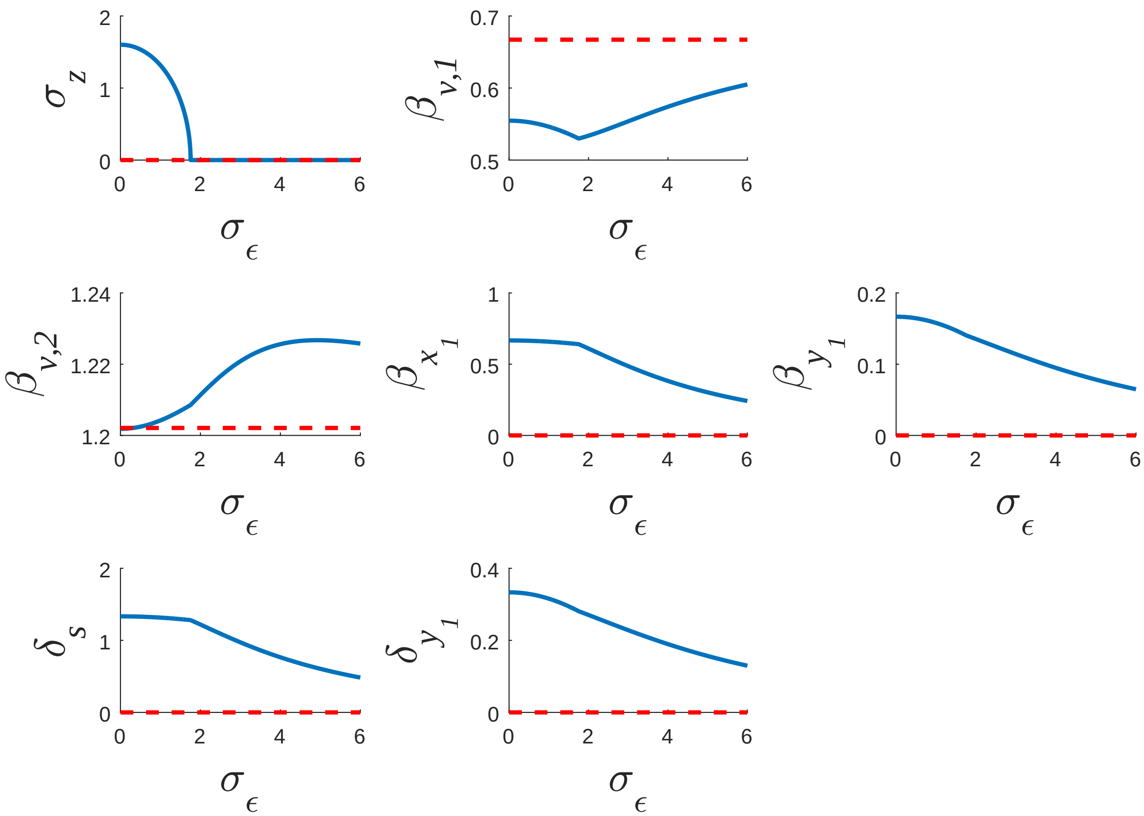
Step 3: If $\frac{\sigma_\varepsilon}{\sigma_u} < \frac{\sqrt{\sqrt{17}-4}}{2}$, employ Proposition 1 to compute a mixed strategy equilibrium.

Figure 2 plots in solid lines the equilibrium trading strategies of the fundamental investor and the back-runner as functions of σ_ε, where we set $\sigma_u = 10$ and $\Sigma_0 = 100$. As a comparison, the dashed lines show corresponding strategies in the standard two-period Kyle model without the back-runner. The first panel confirms that $\sigma_z > 0$ if and only if $\sigma_\varepsilon < 0.175\sigma_u = 1.75$. Also, when $\sigma_\varepsilon < 1.75$, the equilibrium value of σ_z decreases with σ_ε. That is, when there is more *exogenous* noise in the back-runner's signal, the fundamental investor *endogenously* injects less noise into her own period-1 orders. This result points out a new channel—i.e., the amount of noise in the back-runner's signal—that determines whether a mixed strategy equilibrium or a pure strategy one should prevail in a Kyle-type auction game.

The other panels in Figure 2 are also intuitive. For instance, $\beta_{v,1}$ decreases with σ_ε in the mixed strategy regime, but increases with σ_ε in the pure strategy regime. This is because in the mixed strategy regime, as σ_ε increases, the fundamental investor adds less noise z to her order; to avoid revealing too much information to the back-runner, she trades less aggressively on v in period 1. In contrast, in the pure strategy equilibrium, as σ_ε increases, the fundamental investor knows that the back-runner will learn less from her order due to the increased exogenous noise ε, and so she can afford to trade more aggressively in period 1. The intensity $\beta_{v,1}$ with order-flow information is smaller than its counterpart without order-flow information in a standard Kyle model.

An interesting observation is that $\beta_{v,2}$ is hump-shaped in σ_ε, but the peak obtains when σ_ε is substantially above $\sigma_u\sqrt{\sqrt{17}-4}/2$. This is a combination of two effects. First, $\beta_{v,2}$

Figure 2: Implications for Trading Strategies



This figure plots the implications of back-running for trading strategies of the fundamental investor and the back-runner. In each panel, the blue solid line plots the value in the equilibrium of this paper, and the dashed red line plots the value in a standard Kyle economy (i.e., $\sigma_\varepsilon = \infty$). The horizontal axis in each panel is the standard deviation σ_ε of the noise in the back-runner's private signal about the fundamental investor's past order. The other parameters are: $\sigma_u = 10$ and $\Sigma_0 = 100$.

should have a negative relation with $\beta_{v,1}$, as the fundamental investor smoothes her trades across the two periods. Thus, the U-shaped $\beta_{v,1}$ leads to a hump-shaped $\beta_{v,2}$. Second, the fundamental investor also faces competition from the back-runner in the second period, and as σ_ε increases, this competition is less intense, so that the fundamental investor can afford to trade more aggressively on her private information. The second competition effect, adding to the first smoothing effect, implies that the hump-shaped $\beta_{v,2}$ achieves its peak above $\sigma_u\sqrt{\sqrt{17}-4}/2$.

It is straightforward to understand that δ_s decreases with σ_ε: A higher value of σ_ε means that the back-runner's private information s is less precise, and so he trades less aggressively on this information.

3 Implications of Back-Running for Market Quality and Welfare

In this section we discuss the positive and normative implications of back-running, including price discovery, market liquidity, and the trading profits (or losses) of various trader types. Because these measures are proxies for market quality and welfare, our analysis generates important policy implications regarding the use of order-flow informed trading strategies.

We first examine the behavior of positive variables that represent market quality. In the microstructure literature, two leading positive variables are price discovery and market liquidity.[8] Price discovery measures how much information about the asset value v is revealed in prices p_1 and p_2. Given price functions (8) and (9), prices are linear transformations of aggregate order flows y_1 and y_2, and thus the literature has measured price discovery by the market maker's posterior variances of v in periods 1 and 2:

$$\Sigma_1 \equiv Var\left(v|y_1\right) \text{ and } \Sigma_2 \equiv Var\left(v|y_1, y_2\right).$$

A lower Σ_t implies a more informative period-t price about v, for $t \in \{1, 2\}$. Price discovery is important because it helps allocation efficiency by conveying information that is useful for real decisions (see, for example, O'Hara (2003) and Bond, Edmans, and Goldstein (2012)).

In Kyle-type models (including ours), market liquidity is measured by the inverse of Kyle's lambda (λ_1 and λ_2), which are price impacts of trading. A lower λ_t means that the period-t market is deeper and more liquid. One important reason to care about market liquidity is that it is related to the welfare of noise traders, who can be interpreted as investors trading for non-informational, liquidity or hedging reasons that are decided outside the financial markets. In general, noise traders are better off in a more liquid market, because their expected trading loss is $\left(\lambda_1 + \lambda_2\right) \sigma_u^2$ in our economy.

Next, the normative variables are the payoffs of each group of players in the economy, that is, the expected profit $E\left(\pi_{F,1} + \pi_{F,2}\right)$ of the fundamental investor, the expected profit $E(\pi_{B,2})$ of the back-runner, and the expected loss $\left(\lambda_1 + \lambda_2\right) \sigma_u^2$ of noise traders. This approach allows us to discuss who wins and who loses as a result of a particular policy. In practice, investors' trading motives range from fundamental analysis to liquidity shocks (e.g., client withdrawal from mutual funds or hedge funds). Our fundamental investor can be viewed as investors

[8] For example, O'Hara (2003) states that "Markets have two important functions—liquidity and price discovery—and these functions are important for asset pricing."

trading for informational reasons, and noise traders as those trading for liquidity reasons. The back-runner is more in line with broker-dealers or HFTs who employ sophisticated trading technology and may possess superior order-flow information. If the regulator wishes to protect liquidity-driven traders, the welfare of noise traders would be the relevant measure. If the regulator wishes to protect investors who acquire fundamental information, then the informed profit $E\left(\pi_{F,1}+\pi_{F,2}\right)$ would be a relevant measure.

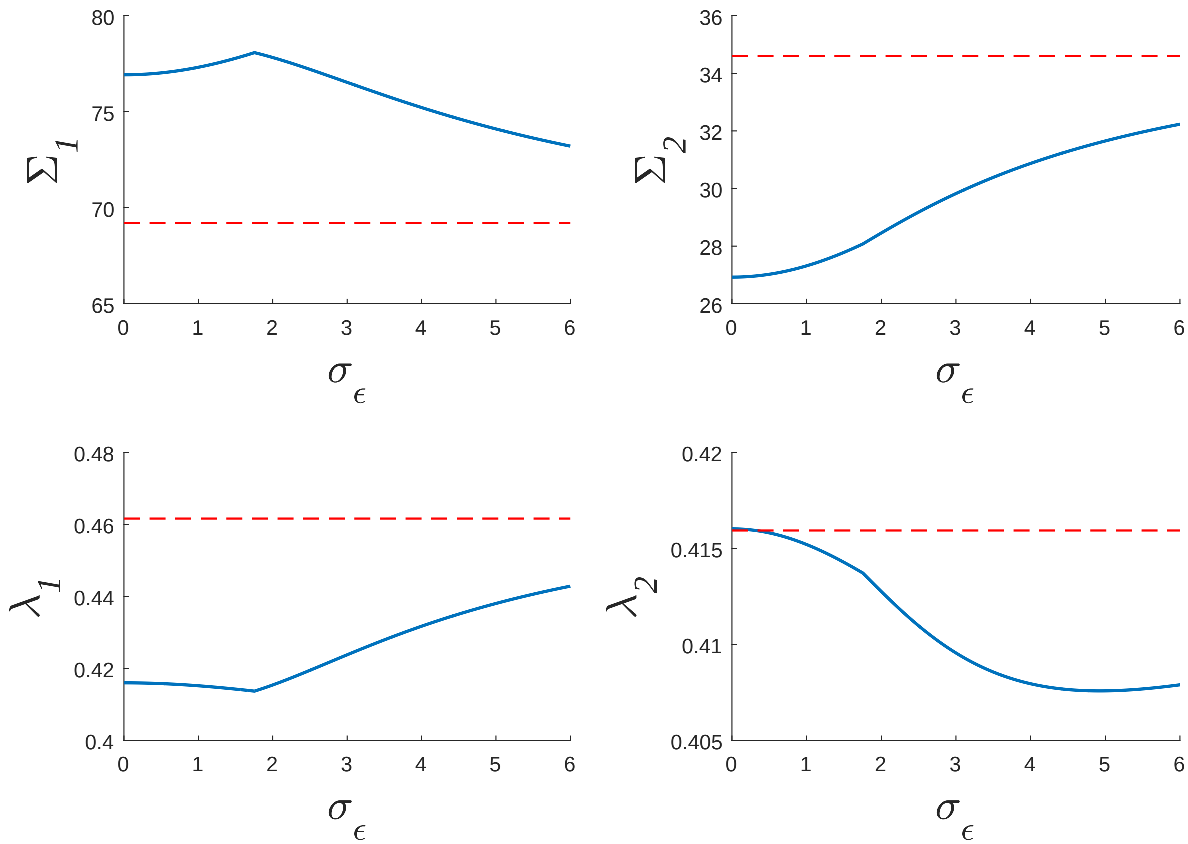
The following proposition gives a comparison between two "extreme" economies: the economy with $\sigma_{\varepsilon}=0$ and the one with $\sigma_{\varepsilon}=\infty$ (i.e. the standard Kyle setting). For instance, the first economy corresponds to one in which back-runners are able to extract very precise information about the past orders submitted by large institutions. The second economy may represent one in which institutional investors manage to hide order-flow information almost completely (or an economy in which back-runners do not participate in the market, due to high technological costs or strict regulations). In the proposition, we have used superscripts "0" and "Kyle" to indicate these two economies.

Proposition 4 (Perfect Order-Flow Information vs. Standard Kyle)**.** *In the two-period setting, the following orderings apply:*

$$\begin{aligned}
\Sigma_1^0 &> \Sigma_1^{Kyle}, \Sigma_2^0 < \Sigma_2^{Kyle}, \\
\lambda_1^0 &< \lambda_1^{Kyle}, \lambda_2^0 > \lambda_2^{Kyle}, \\
E\left(\pi_{F,1}^0\right) &< E\left(\pi_{F,1}^{Kyle}\right), E\left(\pi_{F,2}^0\right) < E\left(\pi_{F,2}^{Kyle}\right) \quad \textit{and} \\
\left(\lambda_1^0+\lambda_2^0\right)\sigma_u^2 &< \left(\lambda_1^{Kyle}+\lambda_2^{Kyle}\right)\sigma_u^2.
\end{aligned}$$

The positive implications in Proposition 4 are in sharp contrast to those presented by Huddart, Hughes, and Levine (2001), although both studies consider a comparison between an economy featuring a mixed strategy equilibrium and a standard Kyle economy. In Huddart, Hughes, and Levine (2001), the market maker perfectly observes the past order placed by an informed trader. They find that market liquidity and price discovery unambiguously improve in both periods of their economy relative to a standard Kyle setting (i.e., their λ_1, λ_2, Σ_1, and Σ_2 are all smaller than the Kyle setting counterparts). In contrast, in our setting, the market maker does not observe the informed fundamental investor's past trade x_1; it is the back-runner who does, with some noise. As a result of the endogenous noise z placed by the fundamental investor in the mixed strategy and her more cautious trading on fundamental information (i.e., a smaller $\beta_{v,1}$), the first-period price discovery is harmed by back-running in our setting (i.e., $\Sigma_1^0 > \Sigma_1^{Kyle}$). The presence of *perfect* information about past order flows also worsens the second period market liquidity relative to the standard

Figure 3: Implications for Positive Variables



This figure plots the market quality implications of back-running. In each panel, the blue solid line plots the value in the equilibrium of this paper, and the dashed red line plots the value in a standard Kyle economy (i.e., $\sigma_\varepsilon = \infty$). The horizontal axis in each panel is the standard deviation σ_ε of the noise in the back-runner's private signal about the fundamental investor's past order. The other parameters are: $\sigma_u = 10$ and $\Sigma_0 = 100$.

Kyle setting (i.e., $\lambda_2^0 > \lambda_2^{Kyle}$). This is again opposite to the effect of publicly revealing the informed orders in period 1, as in Huddart, Hughes, and Levine (2001).

Figures 3 and 4 respectively plot in solid lines the positive and normative implications as we continuously increase σ_ε from 0 to ∞. The other two exogenous parameters are the same as those in Figure 2 ($\sigma_u = 10$ and $\Sigma_0 = 100$). The dashed lines plot the corresponding variables in a standard two-period Kyle model without the back-runner.

In Figure 3, we see that Σ_1 is hump-shaped in σ_ε, with the peak at the cutoff $\sigma_\varepsilon = \sigma_u\sqrt{\sqrt{17} - 4}/2$. The intuition is as follows. In the first period, only the fundamental investor's trade brings information about v into the market. Since her trading sensitivity $\beta_{v,1}$ on fundamental information is U-shaped in σ_ε (see Figure 2), Σ_1 should have the opposite pattern, i.e., hump-shaped. By contrast, Σ_2 monotonically increases with σ_ε in Figure 3. This is because in period 2, both the fundamental investor and the back-runner trade on value-relevant information, and as σ_ε increases, the back-runner's order brings less information about v into the price. Comparing the solid lines to dashed lines, we see that adding the back-runner harms price discovery in period 1 but improves price discovery in period 2.

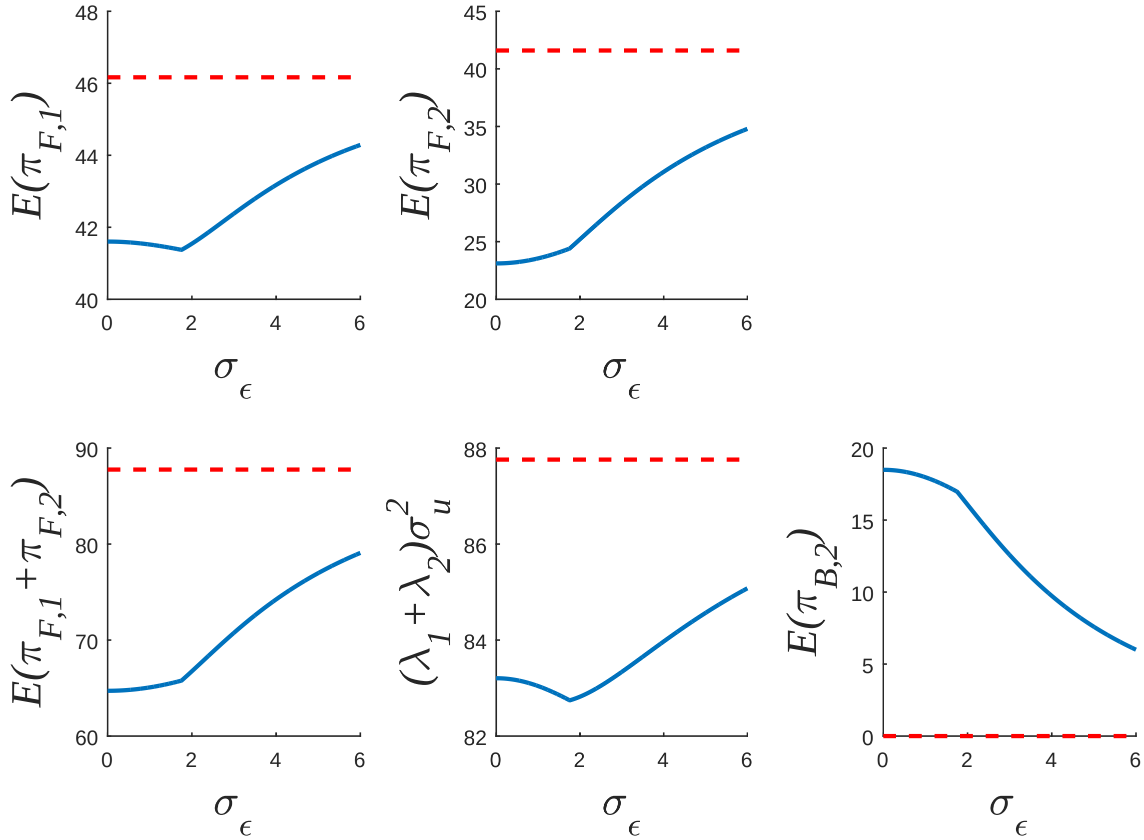
The illiquidity measures in both periods, λ_1 and λ_2, first decrease and then increase with σ_ε. Since adverse selection from the fundamental investor is the sole source of price impact in period 1, it is rather intuitive that λ_1 has a similar U-shape as $\beta_{v,1}$ (see equation (20)). The period-2 illiquidity measure λ_2 is also U-shaped and opposite to the humped-shaped $\beta_{v,2}$, by the first-order condition in period 2 (i.e., $\lambda_2 = \frac{1}{2\beta_{v,2}}$ by (14)).

Comparing the solid lines to dashed lines, we find that back-running generally improves the first-period market liquidity because the fundamental investor trades less aggressively on her private information, but its impact on the second-period market liquidity is ambiguous. Consistent with Proposition 4, back-running worsens the second-period liquidity relative to the standard Kyle setting, if and only if the back-runner's order-flow information is sufficiently precise. (In the neighborhood of $\sigma_\varepsilon = 0$, the solid line is strictly above the dashed line in the plot for λ_2.) This is due to a combination of two effects. First, adding the back-runner introduces competition, which makes the period-2 aggregate order flow reflect more of the fundamental than noise trading. This tends to reduce λ_2. Second, back-running also increases the amount of *private* information, which makes the adverse selection problem faced by the market maker more severe. This generally tends to increase λ_2. When σ_ε is small, the back-runner has very precise private information and the second effect dominates, so that λ_2 is higher than its counterpart in a standard Kyle setting.

The top two panels of Figure 4 plot the fundamental investor's expected profits in the two periods, $E(\pi_{F,1})$ and $E(\pi_{F,2})$. We observe that $E(\pi_{F,2})$ monotonically increases with σ_ε. This result is intuitive: A higher σ_ε means that the fundamental investor faces a less competitive back-runner in period 2, so her period-2 profit is higher on average. The period-1 profit $E(\pi_{F,1})$ first decreases with σ_ε (in the mixed strategy regime) and then increases with σ_ε (in the pure strategy regime). This U-shaped profit pattern is natural given the U-shaped $\beta_{v,1}$ pattern in Figure 2. Comparing the solid lines to dashed lines, we clearly see that back-running lowers the profit of the fundamental investor.

The bottom three panels of Figure 4 present the total profit $E(\pi_{F,1} + \pi_{F,2})$ of the funda-

Figure 4: Implications for Normative Variables



This figure plots the profits of various groups of traders. In each panel, the blue solid line plots the value in the equilibrium of this paper, and the dashed red line plots the value in a standard Kyle economy (i.e., $\sigma_\varepsilon = \infty$). The horizontal axis in each panel is the standard deviation σ_ε of the noise in the back-runner's private signal about the fundamental investor's past order. The other parameters are: $\sigma_u = 10$ and $\Sigma_0 = 100$.

mental investor, the total loss $(\lambda_1 + \lambda_2)\,\sigma_u^2$ of noise traders, and the expected profit $E(\pi_{B,2})$ of the back-runner. All the results are as expected. As σ_ε increases, the back-runner has less precise private information, and thus $E(\pi_{B,2})$ decreases. Meanwhile, a higher σ_ε also implies that the fundamental investor faces less competition from the back-runner, and $E\,(\pi_{F,1} + \pi_{F,2})$ increases. The U-shaped total loss $(\lambda_1 + \lambda_2)\,\sigma_u^2$ of noise traders is a direct result of the U-shaped λ_1 and λ_2 in Figure 3. In general, back-running reduces the loss of noise traders (the entire solid line of $(\lambda_1 + \lambda_2)\sigma_u^2$ lies below the dashed line).

4 Information Acquisition

So far, we have taken the information of the fundamental investor and the back-runner as given. In this section, we explicitly model information acquisition. Besides showing the robustness of our earlier results, this additional step sheds light on questions like "Does back-running discourage acquisition of fundamental information?"

4.1 Setup

We add one period, $t = 0$, before the two-period economy considered in previous sections. At $t = 0$, the fundamental investor decides the amount of fundamental information she acquires, and the back-runner decides the precision of order-flow information he acquires. Specifically, for the fundamental trader, we follow Admati and Pfleiderer (1989), Madrigal (1996), and Bond, Goldstein, and Prescott (2010) and assume that the fundamental investor can pay a cost $C_F(\phi)$ upfront to observe the fundamental value v with probability $\phi \in (0, 1)$. For the back-runner, we follow Verrecchia (1982) and Vives (2008) and assume that the back-runner can pay a cost $C_B\left(\frac{1}{\sigma_\varepsilon^2}\right)$ upfront to observe a signal s of x_1 with precision $\frac{1}{\sigma_\varepsilon^2}$. These information-acquisition decisions are simultaneous. After time 0, the choices of ϕ and σ_ε become public information. In reality, investment in fundamental research, such as hiring analysts, and investment in advanced trading technology, such as high-speed connections to exchanges, are usually observable.

To ensure interior solutions of σ_ε^2 and ϕ, we make the standard technical assumptions: (i) $C_B(\cdot)$ and $C_F(\cdot)$ are increasing and convex; and (ii) $C_B(0) = C_B'(0) = 0, C_B(\infty) = C_B'(\infty) = \infty, C_F(0) = C_F'(0) = 0$, and $C_F(1) = C_F'(1) = \infty$.

For simplicity, we assume that at the beginning of period 1, it becomes public knowledge whether the fundamental investor has successfully observed v. It is a standard assumption in Kyle-type models whether such an (fundamentally) informed investor exists. Then, the subsequent game has two possible outcomes:

1. If the fundamental investor observes v, then the economy is the one that we analyzed in the previous two sections.

2. If the fundamental investor does not observe v, then as an uninformed investor she will not trade in either period. As a result, the back-runner will not trade in period 2, either, despite receiving the signal of the fundamental investor's (zero) order flow. In this case, only noise traders submit orders, and so the price is $p_1 = p_2 = E(v) = p_0$.

4.2 Analysis and Results

Our objective is to find the equilibrium levels of ϕ and σ_ε. They are determined jointly by the period-0 maximization problems of the fundamental investor and the back-runner.

Recall that $\pi_{F,1}$ and $\pi_{F,2}$ denote the realized profits of the fundamental investor in dates 1 and 2, respectively. The fundamental investor's period-0 expected net profit is:

$$\Pi_{F,0} \equiv \phi E\left(\pi_{F,1} + \pi_{F,2}\right) - C_F\left(\phi\right),$$

and her problem is to chose ϕ to maximize $\Pi_{F,0}$, taking her conjectured equilibrium value of σ_ε as given. Because $E\left(\pi_{F,1} + \pi_{F,2}\right)$ does not depend on ϕ, and given the technical assumption on $C_F\left(\phi\right)$, we know that the solution to the fundamental investor's problem is characterized by the first-order condition:

$$E\left(\pi_{F,1} + \pi_{F,2}\right) = C'_F\left(\phi\right).$$

Now we consider the back-runner's information acquisition problem. Recall that

$$\pi_{B,2} \equiv \left(v - p_2\right) d_2$$

is the back-runner's realized period-2 profit. So, his period-0 expected net profit of acquiring order-flow information is:

$$\Pi_{B,0} \equiv \phi E\left(\pi_{B,2}\right) - C_B\left(\frac{1}{\sigma_\varepsilon^2}\right).$$

The back-runner takes the equilibrium value ϕ as given and chooses σ_ε to maximize Π_0^B.

The back-runner's choice of σ_ε affects $E\left(\pi_{B,2}\right)$ through its effect on the equilibrium strategies, $\sigma_z, \beta_{v,1}, \beta_{v,2}, \beta_{x_1}, \beta_{y_1}, \delta_s, \delta_{y_1}, \lambda_1$ and λ_2. Specifically, we can compute

$$E\left(\pi_{B,2}\right) = \lambda_2\left[\left(\delta_s - \delta_{y_1}\right)^2 \beta_{v,1}^2 \Sigma_0 + \left(\delta_s - \delta_{y_1}\right)^2 \sigma_z^2 + \delta_s^2 \sigma_\varepsilon^2 + \delta_{y_1}^2 \sigma_u^2\right],$$

and hence

$$\Pi_{B,0} = \phi\lambda_2\left[\left(\delta_s - \delta_{y_1}\right)^2 \beta_{v,1}^2 \Sigma_0 + \left(\delta_s - \delta_{y_1}\right)^2 \sigma_z^2 + \delta_s^2 \sigma_\varepsilon^2 + \delta_{y_1}^2 \sigma_u^2\right] - C_B\left(\frac{1}{\sigma_\varepsilon^2}\right).$$

There is an important complication in the solution to the back-runner's information-acquisition problem. Although this problem has an interior solution, as ensured by the cost function $C_B(\cdot)$, the optimal choice of σ_ε cannot in general be guaranteed by setting the first-order derivative to zero. This is because whether the equilibrium has a mixed strategy or a pure strategy (used by the fundamental investor) depends on σ_ε. As σ_ε decreases and drops below the threshold value of $(\sqrt{\sqrt{17} - 4}/2)\sigma_u$, the equilibrium switches from pure strategy to mixed strategy, giving rise to a kink in $E\left(\pi_{B,2}\right)$. If the optimal value of σ_ε occurs at the kink, the first-order condition is characterized by two inequalities rather than an equality. (This complication does not apply to the fundamental investor's problem.)

To solve the equilibrium explicitly and numerically, we need explicit functional forms of

C_B and C_F. Following Vives (2008), we choose the following parametrization:

$$C_B\left(\frac{1}{\sigma_\varepsilon^2}\right) = k_B\left(\frac{1}{\sigma_\varepsilon^2}\right)^{h_B} = k_B\sigma_\varepsilon^{-2h_B},$$

$$C_F(\phi) = k_F\left(\frac{\phi}{1-\phi}\right)^{h_F},$$

where

$$k_B > 0, k_F > 0, h_B > 1 \text{ and } h_F > 1.$$

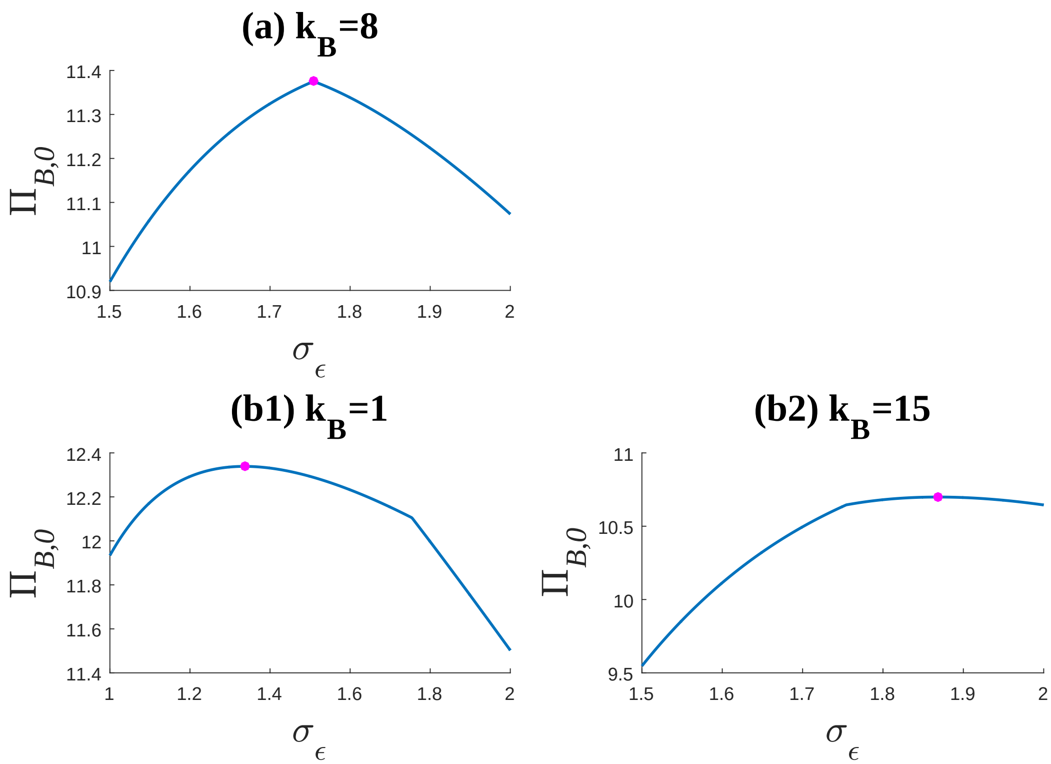
We will conduct comparative statics with respect to parameter k_B, which is taken as a proxy for the cost of acquiring order-flow information. A larger k_B means a higher cost.

In order to gain better intuition of the comparative statics, it is useful to first illustrate the kink in $\Pi_{B,0}$. Figure 5 plots the profit function $\Pi_{B,0}$ against σ_ε, for $k_B \in \{1, 8, 15\}$, in the three panels. In each panel, ϕ is set to its equilibrium value corresponding to the particular k_B and does not vary with σ_ε, since this value of ϕ is the belief of the back-runner at the information-acquisition stage. But for each σ_ε, other equilibrium variables in periods 1 and 2 are determined according to Propositions 1 and 2 for this particular σ_ε (and the fixed equilibrium value of ϕ), because at the information-acquisition stage, the back-runner takes into account how the fundamental investor and the market maker react in future periods. As in earlier figures, we set $\sigma_u = 10$ and $\Sigma_0 = 100$. We also set $k_F = 1$ and $h_F = h_B = 2$.

In Panel (a), where $k_B = 8$, the optimal σ_ε occurs exactly at the kink. In Panels (b1) and (b2), where $k_B = 1$ and $k_B = 15$ respectively, the optimal values of σ_ε are found in the smooth regions. Intuitively, if the information-acquisition cost k_B is very high or very low, the unconstrained optimal σ_ε—the solution without considering the equilibrium switch—is sufficiently far away from the threshold $(\sqrt{\sqrt{17}-4}/2)\sigma_u$, so the switch in equilibrium does not bind, as in Panels (b1) and (b2). If, however, k_B takes an intermediary value, the nature of equilibrium depends heavily on σ_ε. In the mixed strategy region of Panel (a), i.e. if $\sigma_\varepsilon < (\sqrt{\sqrt{17}-4}/2)\sigma_u$, the back-runner prefers to acquire less precise information because the fundamental investor injects noise anyway; but in the pure strategy region of Panel (a), i.e. if $\sigma_\varepsilon \geq (\sqrt{\sqrt{17}-4}/2)\sigma_u$, the back-runner prefers more precise information because the fundamental investor does not inject any noise. The result is that the unique maximum of $\Pi_{B,0}$ is obtained when σ_ε is exactly at the threshold $(\sqrt{\sqrt{17}-4}/2)\sigma_u$. As we see shortly, this corner solution leads to the stickiness in the responses of equilibrium outcomes to changes in k_B.

Now we proceed with describing the comparatives statics. The variables of interest include:

Figure 5: Illustration of Possible Kink in $\Pi_{B,0}$



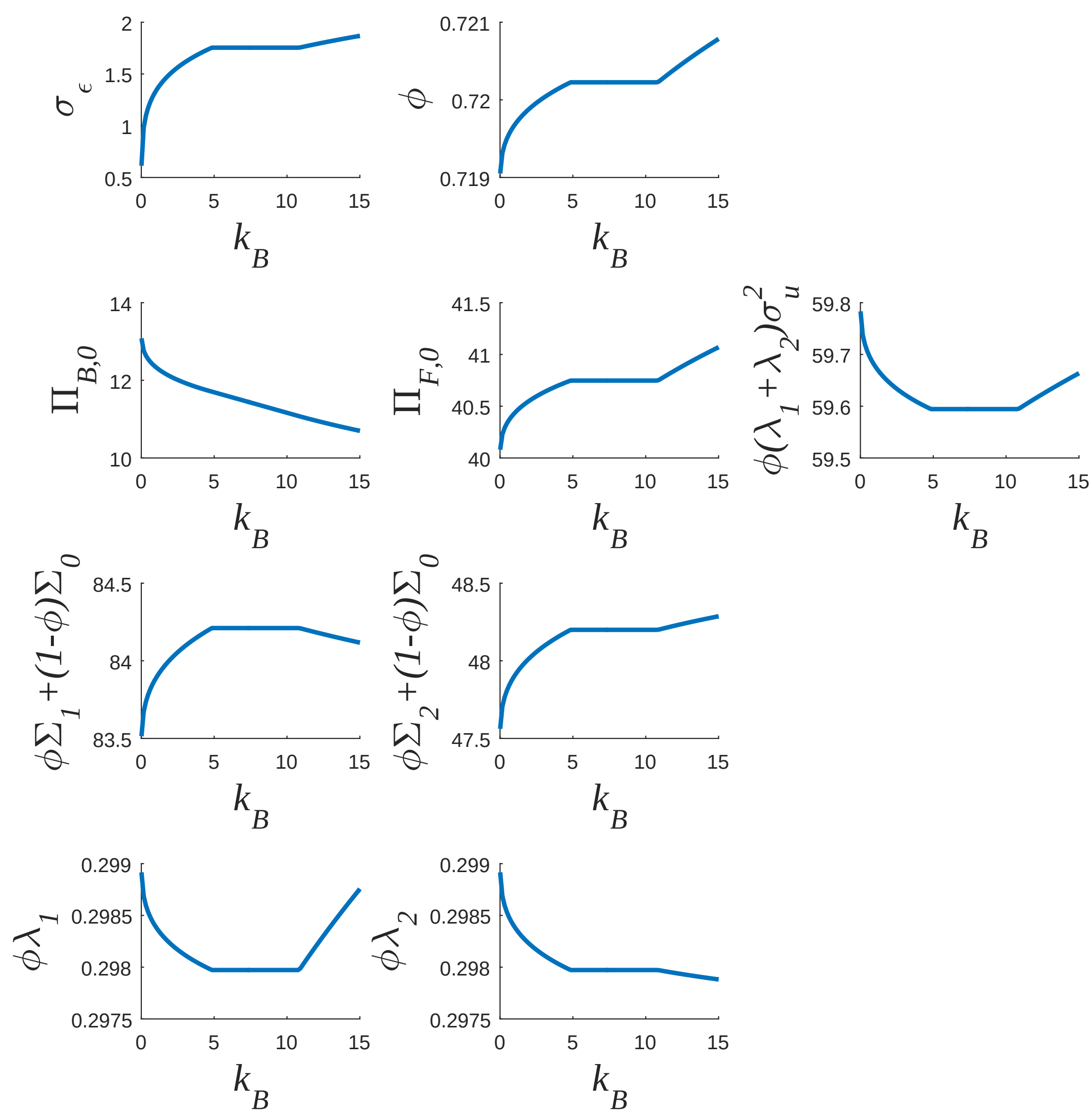
This figure plots $\Pi_{B,0}$ against σ_ε for three values of k_B. In each panel, ϕ is set to its equilibrium value corresponding to the particular k_B and does not vary with σ_ε. For each σ_ε, other equilibrium variables in periods 1 and 2 are optimized to this particular σ_ε. The red dot is the global maximum. Other parameters: $\sigma_u = 10, \Sigma_0 = 100, k_F = 1$, and $h_F = h_B = 2$.

- Equilibrium values of $\phi, \sigma_\varepsilon, \sigma_z, \beta_{v,1}, \beta_{v,2}, \beta_{x_1}, \beta_{y_1}, \delta_s, \delta_{y_1}, \lambda_1$ and λ_2;
- Equilibrium profits: $\Pi_{F,0}$ and $\Pi_{B,0}$, and the expected cost of noise traders $\Pi_{F,0} + \Pi_{B,0}$;
- Price discovery: $\phi\Sigma_1 + (1 - \phi)\,\Sigma_0$ for period 1 and $\phi\Sigma_2 + (1 - \phi)\,\Sigma_0$ for period 2;
- Illiquidity: $\phi\lambda_1 + (1 - \phi)\,0 = \phi\lambda_1$ for period 1 and $\phi\lambda_2 + (1 - \phi)\,0 = \phi\lambda_2$ for period 2.

Here, we follow the literature and measure average price discovery and illiquidity, where λ_1 and λ_2 are defined in Propositions 1 and 2, and Σ_1 and Σ_2 are defined at the beginning of Section 3.

Figure 6 plots the implications of changes in information acquisition cost k_B for information-acquisition decisions, profits of various groups of traders, price discovery, and market illiq-

Figure 6: Equilibrium Strategies and Implications of Information Acquisition



This figure plots the equilibrium levels of information acquisition, expected profits of various parties, price discovery, and market illiquidity, as functions of k_B. Other parameters: $\sigma_u = 10, \Sigma_0 = 100, k_F = 1$, and $h_F = h_B = 2$.

uidity.

An interesting and salient pattern is that all but one of these variables are entirely irresponsive to changes in k_B when k_B is in an intermediate range. As discussed earlier, in this range, the optimal σ_ε is always equal to $(\sqrt{\sqrt{17-4}/2})\sigma_u$ regardless of k_B. As a result, the equilibrium has zero sensitivity to k_B, leading to the flat parts of equilibrium variables.

Moreover, we observe that a lower k_B weakly reduces ϕ, which implies that technology improvement in processing order-flow information reduces investment in fundamental information (top row of Figure 6). A lower cost of acquiring order-flow information leads to a higher profit of the back-runner but a lower profit of the fundamental investor. The loss of noise traders, the period-1 price discovery, and the period-1 market liquidity are all non-monotone in k_B. This last result mirrors the patterns in Figures 3 and 4 that these variables are also non-monotone in σ_ε.

Overall, results of the previous sections are robust to information acquisition. A unique and novel prediction with information acquisition is that equilibrium outcomes can be insensitive to the cost of order-flow information. This insensitivity is the consequence of the switch between a pure strategy equilibrium and a mixed strategy one.

5 Empirical Relevance

This section discusses how our theory helps interpret recent evidence on the behavior of high-frequency traders (HFTs). Although back-running is not conducted exclusively by HFTs, HFTs stand out as the most relevant application in today's markets. Mapping the model into reality, we can approximately view the fundamental investor as (some) institutional investors, the back-runner as order-anticipating HFTs, the market maker as a mix of market-making HFTs and human traders posting limit orders, and the noise trader as a mix of retail and other institutional investors. To be conservative, we shall use the weaker "correlation," rather than "causality," interpretation of evidence discussed in this section.

5.1 HFT and Institutional Investors

van Kervel and Menkveld (2015) study the trading behaviors of HFTs when large institution investors execute orders in the Swedish equity market. Data from NASDAQ-OMX, the main equity exchange in Nordic markets, disclose the identifiers of exchange members, including HFTs. In addition, they use a proprietary dataset that contains the detailed transaction records of four institutional investors. Institutions typically split large orders into smaller pieces and execute them over time. Matching these two data sources, van Kervel and Menkveld (2015) investigate whether HFTs take the opposite side of institutional investor order flows ("lean against the wind") or trade in the same direction ("go with the wind").

van Kervel and Menkveld (2015) find that HFTs lean against the wind for the first six hours of institutional buy orders and for the first two hours of institutional sell orders. This initial behavior is consistent with the market-making activity of HFTs. Going back to our model, the first period can be viewed as an abstraction of this initial phase when the back-runner does not trade and when order flows are handled by the market maker (e.g. market-making HFT strategies and human traders).

Interestingly, HFTs reverse course and trade in the same direction as the institutional order if the order lasts more than six hours for buys and more than two hours for sells. For those orders HFTs' inventories eventually end up in the same direction as the institutions. This behavior is precisely predicted by our theory: HFTs learn valuable information from institutions' past order flows and eventually compete with them. Moreover, van Kervel and Menkveld (2015) find that institutions' implementation shortfall[9] is higher if HFTs go in the same direction as institutions than if HFTs go opposite with institutions. Again, this piece of evidence supports our theory that once the back-runner starts to compete with the fundamental investor, the price converges to the fundamental value faster on average; this faster convergence is manifested as a higher effective transaction cost for the (informed) fundamental investor.

Furthermore, van Kervel and Menkveld (2015) test our theory of back-running against the predatory-trading theory of Brunnermeier and Pedersen (2005). In Brunnermeier and Pedersen (2005), the predator starts trading at the same time as the prey in the same direction, before reversing course, and the price impact is transitory because the prey's trades are liquidity-driven. In our theory, the back-runner starts by learning from order-flow information, and the eventual price impact is permanent because the institution's trades are information-driven. As van Kervel and Menkveld (2015) point out, the fact that HFTs behave like market makers for hours before reversing course directly supports our theory.

While the Swedish data are the most transparent, HFT studies in U.S. and Canada find broadly similar results. Using the NASDAQ HFT data, Tong (2015) finds that an increase in HFT activities is associated with a higher implementation-shortfall cost of institutions. In the Canadian equity market, Korajczyk and Murphy (2014) find that implementation shortfalls are higher if HFTs take more liquidity, controlling for the level of activities of

[9]The implementation shortfall measures the extent to which the average transaction price of a large order is worse than the price at the start of the execution. For example, if an institution's average purchase price is \$10.05 and the price at the beginning of execution is \$10.00, the implementation shortfall is 50 basis points ($10.05/10.00 - 1 = 0.5\%$). In Kyle-type models, the implementation shortfall of the informed trader is positive in expectation because his trades gradually reveal information and push the price in the adverse direction (for the informed trader).

HFTs and designated market makers.

It should be stressed that the above evidence does not conflict with earlier research findings that "HFT and automated, competing trading venues have substantially improved market liquidity and reduced trading costs for all investors" (see Jones (2013), who provides a detailed survey of HFT studies up to March 2013). To see this, note that many HFTs enter as more efficient market makers than human ones, and the resulting competition reduces investors' transaction costs, especially for small orders. It is the largest institutional orders that are exposed to back-running strategies. It is our understanding that all HFT studies, including those suggesting a negative impact of HFT on institutional execution performance, fully acknowledge that transaction costs have declined substantially in the past ten years when HFTs have been playing an increasingly important role in equity markets. Similarly, many proponents of HFTs acknowledge that not all HFT strategies help investors. Back-running merits an in-depth study because it is a salient example of controversial HFT practice.

5.2 HFT and Price Discovery

An important implication from our analysis of market quality is that the fundamental investor's reduced trading intensity and camouflage by injecting noise into order flows delay price discovery. Recent evidence from Weller (2015) supports this prediction. Using data from SEC's Market Information Data Analytics System (MIDAS), Weller (2015) links proxies of algorithmic trading to measures of price discovery before earnings announcements. His proxies to algorithmic trading include odd lots, trade-to-order ratio, cancellation-to-trade ratio, and trade size. His measure of price discovery is defined as the "jump ratio" $\Delta p^{(T-1,T+2)}/\Delta p^{T-22,T+2}$, where T is the earnings announcement date and $\Delta p^{(k_1,k_2)}$ is the stock return from day k_1 to day k_2 after adjusting for Fama-French three factors. A larger jump ratio means that a smaller fraction of price change happens before earnings announcement, i.e., a worse price discovery.

The main finding of Weller (2015) is that more active algorithmic trading is associated with a larger jump ratio, hence worse price discovery before earnings announcements. This evidence supports our model prediction that back-running delays price discovery.

We emphasize that the theoretical prediction that back-running delays price discovery is not inconsistent with the existing literature on HFT and price discovery. For example, Brogaard, Hendershott, and Riordan (2014) find that HFTs "facilitate price efficiency by trading in the direction of permanent price changes and in the opposite direction of transitory pricing errors," although HFTs' information advantage lasts only for a few seconds. The directions of HFT trades are correlated with public information such as macroeconomic data releases and limit order book imbalances. Their evidence suggests that HFTs' information advantage could come from their superior ability to process various kinds of public information. Since the back-runner in our model parses public order flows better than others, the back-runner's behavior in our theory is in fact highly consistent with the evidence from Brogaard, Hendershott, and Riordan (2014). Precisely because HFTs are good at parsing public information, including order flows, the fundamental investor in our model releases less information to prices.

Separately, Hirschey (2013) documents that aggressive HFT orders predict non-HFT orders in the immediate future, and he interprets this result as consistent with the hypothesis that HFTs make money partly "by identifying patterns in trade and order data that allow them to anticipate and trade ahead of other investors' order flow"—which, again, is precisely back-running.

6 Conclusion

Order-flow informed trading is a salient part of modern financial markets. This type of trading strategies, such as order anticipation, often starts with no innate trading motive, but instead seeks and exploits information from other investors' past order flows. We refer to such strategies as back-running. While back-running has long existed in financial markets, its latest incarnation in certain high-frequency trading strategies caused renewed and severe concerns among investors and regulators.

In this paper we study the strategic interaction between back-runners and fundamental informed investors. In our two-period model, which is based on Kyle (1985), a back-runner observes, *ex post* and potentially with noise, the executed trades of the informed investor in period 1. The informed order flow thus provides a signal to the back-runner regarding the asset fundamental value. Using this information, the back-runner competes with the informed investor in period 2. While simple, this model structure parsimoniously captures the key idea of back-running.

If the back-runner's signal is sufficiently precise, the fundamental investor hides her information by endogenously adding noise into her period-1 order flow, leading to a mixed strategy equilibrium. The more precise is the order-flow signal, the more volatile is the added noise. As the back-runner's signal becomes sufficiently imprecise, the equilibrium switches to a pure strategy one, in which the fundamental investor adds no endogenous noise in her order flows. We prove uniqueness of equilibrium under natural conditions. The characterization of the equilibria, in particular the endogenous switch between a mixed strategy equilibrium and a pure strategy one, is the first main contribution of this paper.

Our second main contribution is to identify the effects of back-running on market quality. Because the fundamental investor trades more cautiously and potentially adds noise into her period-1 orders, the presence of the back-runner harms price discovery in the first period. In the second period, however, price discovery is improved because of competition. Effects on market liquidity, measured by the inverse of Kyle's lambda, are mixed: Liquidity improves in the first period but can either improve or worsen in the second period.

Our main results are robust to endogenous information acquisition. Additionally, we find that a lower cost of acquiring order-flow information reduces the fundamental investor's incentive to acquire fundamental information.

Recent evidence on high-frequency trading supports our theoretical results. Since back-running is one of high-frequency trading strategies, our results should not be interpreted as a one-size-fits-all characterization of all HFTs, especially market-making HFTs. That said, our results are still highly relevant because back-running is arguably the most controversial HFT practice and continues to cause concerns among investors and regulators.

While our model is made as simple and parsimonious as possible, a couple of extensions could be entertained. First, one could allow multiple informed investors and multiple back-runners. We expect that the additional informed investors create a free-riding problem and weaken the incentives to add noise to their period-1 strategies, but the additional back-runners increase the risk of information leakage for informed investors and encourage them to add more noise in the first period. A second possible extension is to write a dynamic back-running model with more than two periods. A challenge of this extension is history-dependence, that is, strategies in period t can potentially depend on variables in periods 1, 2, ..., $t-1$. These extensions, while potentially interesting, are unlikely to change our main results, and we leave them for future research.

Appendix

A List of Model Variables

Variables	Description
Random Variables	
v	Asset liquidation value at the end of period 2, $N(p_0, \Sigma_0)$
x_1, x_2	Orders placed by the fundamental investor in periods 1 and 2
z	Noise component in the period-1 order x_1 of the fundamental investor
d_2	Order placed by the back-runner in period 2
s, ε	Signal observed by the back-runner, and its noise component
u_1, u_2	Noise trading in periods 1 and 2
y_1, y_2	Aggregate order flows in periods 1 and 2
p_1, p_2	Asset prices in periods 1 and 2
$\pi_{F,1}, \pi_{F,2}$	Fundamental investor's profits attributable to trades in periods 1 and 2
$\pi_{B,2}$	Back-runner's profit in period 2
σ_ε	Noise in back-runner's signal of x_1
ϕ	Fundamental investor's probability of observing v (only in Section 4)
$\Pi_{F,0}$	$\phi E(\pi_{F,1} + \pi_{F,2}) - C_F(\phi)$ (only in Section 4)
$\Pi_{B,0}$	$\phi E(\pi_{B,2}) - C_B\left(\frac{1}{\sigma_\varepsilon^2}\right)$ (only in Section 4)
Deterministic Variables	
p_0, Σ_0	Prior mean and variance of the asset value
σ_u^2	Variance of noise trading in periods 1 and 2
σ_z	Standard deviation of the noise component z in the period-1 order x_1 placed by the fundamental investor
Σ_1, Σ_2	Posterior variance of the asset value in periods 1 and 2 (i.e., $Var(v\|y_1)$ and $Var(v\|y_1, y_2)$)
$C_F(\phi)$	Fundamental investor's cost to observe v with probability ϕ
$C_B(1/\sigma_\varepsilon^2)$	Back-runner's cost of observing a signal of x_1 with precision $1/\sigma_\varepsilon^2$
Strategy Summary	
$\beta_{v,1}$	$x_1 = \beta_{v,1}(v-p) + z$
$\beta_{v,2}, \beta_{x_1}, \beta_{y_1}$	$x_2 = \beta_{v,2}(v-p) - \beta_{x_1}x_1 + \beta_{y_1}y_1$
δ_s, δ_{y_1}	$d_2 = \delta_s s - \delta_{y_1}y_1$
λ_1	$p_1 = p_0 + \lambda_1 y_1$, with $y_1 = x_1 + u_1$
λ_2	$p_2 = p_1 + \lambda_2 y_2$, with $y_2 = x_2 + d_2 + u_2$

B Proofs

B.1 Proof of Equation (10)

Define $\sigma_x^2 \equiv Var(x_1) = \beta_{v,1}^2 \Sigma_0 + \sigma_z^2$. Direct computation shows

$$E(v|s, y_1) - E(v|y_1) = \frac{\beta_{v,1}\Sigma_0 \sigma_\varepsilon^{-2}}{\sigma_x^2 \left(\sigma_x^{-2} + \sigma_\varepsilon^{-2} + \sigma_u^{-2}\right)} \left(s - \frac{\sigma_x^2}{\sigma_u^2 + \sigma_x^2} y_1\right).$$

Thus, it suffices to show that

$$\frac{\delta_{y_1}}{\delta_s} = \frac{\sigma_x^2}{\sigma_u^2 + \sigma_x^2} \tag{B1}$$

holds in equilibrium, in order for d_2 in equation (7) to admit a form given by equation (10).

By equation (18), we have:

$$\frac{\beta_{v,1}\Sigma_0}{\delta_s \lambda_2} = \frac{\sigma_x^2 \left[4\left(\sigma_x^{-2} + \sigma_\varepsilon^{-2} + \sigma_u^{-2}\right) - \sigma_\varepsilon^{-2}\right]}{\sigma_\varepsilon^{-2}}. \tag{B2}$$

Plugging the expression of $\lambda_1 = \frac{\beta_{v,1}\Sigma_0}{\sigma_x^2 + \sigma_u^2}$ (i.e. equation (20)) into equation (19) yields

$$\frac{\delta_{y_1}}{\delta_s} = \frac{\beta_{v,1}\Sigma_0}{\delta_s \lambda_2} \frac{1}{3\left(\sigma_x^2 + \sigma_u^2\right)} - \frac{4\sigma_\varepsilon^2}{3\sigma_u^2}. \tag{B3}$$

Inserting equation (B2) into (B3) and simplifying, we have equation (B1).

B.2 Proof of Proposition 1

A mixed strategy equilibrium is characterized by nine parameters, $\sigma_z, \beta_{v,1}, \beta_{v,2}, \beta_{y_1}, \beta_{x_1}, \delta_{y_1}, \delta_s$, λ_1, and λ_2. These parameters are jointly determined by a system consisting of nine equations (given by (14), (18), (19), (20), (21), and (23)) as well as one SOC ($\lambda_2 > 0$ given by (13)). Note that by equation (23), δ_s is already known, and also $\lambda_1 = \lambda_2$ degenerates to one parameter, denoted by λ. So, the system characterizing a mixed strategy equilibrium essentially has six unknowns. To solve this system, we first simplify it to a 3-equation system in terms of $(\lambda, \beta_{v,1}, \sigma_z)$ and then solve this new system analytically.

Given that δ_s is known, parameter δ_{y_1} is also known by (19). Also, once λ is solved, the three equations in (14) will yield solutions of $\beta_{v,2}, \beta_{x_1}$, and β_{y_1}. Thus, the three equations left to compute $(\lambda, \beta_{v,1}, \sigma_z)$ are given by equations (18), (20) and (21). To solve this 3-equation system, we first express $\beta_{v,1}$ and λ as functions of σ_z, and then solve the single equation of σ_z.

By (18) and noting that $\lambda \equiv \lambda_1 = \lambda_2$, we have

$$\lambda = \frac{1}{\delta_s} \frac{\frac{\sigma_\varepsilon^{-2}}{\left(\beta_{v,1}^2 \Sigma_0 + \sigma_z^2\right)^{-1} + \sigma_\varepsilon^{-2} + \sigma_u^{-2}}}{4 - \frac{\sigma_\varepsilon^{-2}}{\left(\beta_{v,1}^2 \Sigma_0 + \sigma_z^2\right)^{-1} + \sigma_\varepsilon^{-2} + \sigma_u^{-2}}} \frac{\beta_{v,1}\Sigma_0}{\beta_{v,1}^2 \Sigma_0 + \sigma_z^2}.$$

Combining the above equation with (20) and the expression $\delta_s = \frac{\frac{4}{3}}{1 + \frac{4\sigma_\varepsilon^2}{3\sigma_u^2}}$, we can compute

$$\beta_{v,1}^2 = \frac{\sigma_u^4 - 3\sigma_u^2\sigma_z^2 - 4\sigma_u^2\sigma_\varepsilon^2 - 4\sigma_z^2\sigma_\varepsilon^2}{3\Sigma_0\sigma_u^2 + 4\Sigma_0\sigma_\varepsilon^2}. \tag{B4}$$

Equation (B4) puts an restriction on the endogenous value of σ_z, i.e., $\sigma_u^4 - 3\sigma_u^2\sigma_z^2 - 4\sigma_u^2\sigma_\varepsilon^2 - 4\sigma_z^2\sigma_\varepsilon^2 > 0$, which can be shown to hold in equilibrium.

By (20) and (B4), we can express λ^2 as a function of σ_z^2 as follows:

$$\lambda^2 = \frac{\frac{\sigma_u^4 - 3\sigma_u^2\sigma_z^2 - 4\sigma_u^2\sigma_\varepsilon^2 - 4\sigma_z^2\sigma_\varepsilon^2}{3\Sigma_0\sigma_u^2 + 4\Sigma_0\sigma_\varepsilon^2}\Sigma_0^2}{\left(\frac{\sigma_u^4 - 3\sigma_u^2\sigma_z^2 - 4\sigma_u^2\sigma_\varepsilon^2 - 4\sigma_z^2\sigma_\varepsilon^2}{3\Sigma_0\sigma_u^2 + 4\Sigma_0\sigma_\varepsilon^2}\Sigma_0 + \sigma_z^2 + \sigma_u^2\right)^2}. \tag{B5}$$

Inserting $\delta_s = \frac{\frac{4}{3}}{1 + \frac{4\sigma_\varepsilon^2}{3\sigma_u^2}}$ into equation (21) and further simplification yield

$$\begin{aligned} &\left(\begin{array}{c} \left(52\Sigma_0\sigma_u^6 + 160\Sigma_0\sigma_u^4\sigma_\varepsilon^2 + 64\Sigma_0\sigma_u^2\sigma_\varepsilon^4\right)\beta_{v,1}^2 \\ + \left(36\sigma_u^8 + 52\sigma_u^6\sigma_z^2 + 160\sigma_u^6\sigma_\varepsilon^2 + 160\sigma_u^4\sigma_z^2\sigma_\varepsilon^2 + 64\sigma_u^4\sigma_\varepsilon^4 + 64\sigma_u^2\sigma_z^2\sigma_\varepsilon^4\right) \end{array} \right)\lambda^2 \\ &= \left(9\Sigma_0\sigma_u^6 + 9\Sigma_0\sigma_u^4\sigma_z^2 + 24\Sigma_0\sigma_u^4\sigma_\varepsilon^2 + 24\Sigma_0\sigma_u^2\sigma_z^2\sigma_\varepsilon^2 + 16\Sigma_0\sigma_u^2\sigma_\varepsilon^4 + 16\Sigma_0\sigma_z^2\sigma_\varepsilon^4\right). \end{aligned}$$

Inserting equations (B4) and (B5) into the above equation, we can compute

$$\sigma_z^2 = \frac{\sigma_u^2\left(\sigma_u^2 + 4\sigma_\varepsilon^2\right)\left(\sigma_u^4 - 16\sigma_\varepsilon^4 - 32\sigma_u^2\sigma_\varepsilon^2\right)}{\left(3\sigma_u^2 + 4\sigma_\varepsilon^2\right)\left(13\sigma_u^4 + 16\sigma_\varepsilon^4 + 40\sigma_u^2\sigma_\varepsilon^2\right)}, \tag{B6}$$

which gives the expression of σ_z in Proposition 1.

In order for equation (B6) to indeed construct a mixed strategy equilibrium, we need

$$\sigma_z^2 = \frac{\sigma_u^2\left(\sigma_u^2 + 4\sigma_\varepsilon^2\right)\left(\sigma_u^4 - 16\sigma_\varepsilon^4 - 32\sigma_u^2\sigma_\varepsilon^2\right)}{\left(3\sigma_u^2 + 4\sigma_\varepsilon^2\right)\left(13\sigma_u^4 + 16\sigma_\varepsilon^4 + 40\sigma_u^2\sigma_\varepsilon^2\right)} > 0 \Leftrightarrow \frac{\sigma_\varepsilon^2}{\sigma_u^2} < \frac{\sqrt{17}}{4} - 1.$$

Also, inserting equation (B6) into equation (B4), we see that (B4) is always positive. Finally, by equation (20) and $\lambda_2 = \lambda_1$, we know $\lambda_2 > 0$, i.e., the SOC is satisfied. Thus, when $\frac{\sigma_\varepsilon^2}{\sigma_u^2} < \frac{\sqrt{17}}{4} - 1$, the expression of σ_z^2 in equation (B6) indeed constructs a mixed strategy equilibrium.

Clearly, if $\frac{\sigma_\varepsilon^2}{\sigma_u^2} \geq \frac{\sqrt{17}}{4} - 1$, then the solved σ_z^2 would be non-positive in (B6), which implies the non-existence of a linear mixed strategy equilibrium.

B.3 Proof of Proposition 2

For a pure strategy equilibrium, we have $\sigma_z = 0$ and need to compute eight parameters, $\beta_{v,1}, \beta_{v,2}, \beta_{y_1}, \beta_{x_1}, \delta_s, \delta_{y_1}, \lambda_1$, and λ_2. These parameters are determined by equations (14), (18), (19), (20), (21), and (24), together with two SOC's, (13) and (25). In particular, after setting $\sigma_z = 0$, we can simplify equations (18), (20), and (21) as follows:

$$\delta_s = \frac{\frac{\sigma_\varepsilon^{-2}}{\left(\beta_{v,1}^2\Sigma_0\right)^{-1}+\sigma_\varepsilon^{-2}+\sigma_u^{-2}}}{4-\frac{\sigma_\varepsilon^{-2}}{\left(\beta_{v,1}^2\Sigma_0\right)^{-1}+\sigma_\varepsilon^{-2}+\sigma_u^{-2}}}\frac{1}{\lambda_2\beta_{v,1}}, \tag{B7}$$

$$\lambda_1 = \frac{\beta_{v,1}\Sigma_0}{\beta_{v,1}^2\Sigma_0+\sigma_u^2}, \tag{B8}$$

$$\lambda_2 = \frac{\left(\frac{1}{2\lambda_2}+\frac{\delta_s}{2}\beta_{v,1}\right)\frac{1}{\Sigma_0^{-1}+\beta_{v,1}^2\sigma_u^{-2}}}{\left(\frac{1}{2\lambda_2}+\frac{\delta_s}{2}\beta_{v,1}\right)^2\frac{1}{\Sigma_0^{-1}+\beta_{v,1}^2\sigma_u^{-2}}+\delta_s^2\sigma_\varepsilon^2+\sigma_u^2}. \tag{B9}$$

Note that equation (B8) is the expression of λ_1 in Proposition 2.

The idea to compute the system characterizing a pure strategy equilibrium is to simplify it to a system in terms of $(\lambda_1, \lambda_2, \beta_{v,1}, \delta_s)$ and then characterize this simplified system as a single equation of $\beta_{v,1}$.

If we know $(\lambda_1, \lambda_2, \delta_s)$, then δ_{y_1} is known by equation (19), and β_{y_1}, $\beta_{v,2}$, and β_{x_1} are known by equation (14). Thus, the four unknowns $(\lambda_1, \lambda_2, \beta_{v,1}, \delta_s)$ are determined by the remaining four equations, (24) and (B7)–(B9), and the two SOC's, (13) and (25).

Now, we simplify this four-equation system as a single equation of $\beta_{v,1}$. The idea is to express λ_1, $\lambda_2\delta_s$ and λ_2 as functions of $\beta_{v,1}$, and then insert these expressions into equation (24). By (B7),

$$\lambda_2\delta_s = \frac{\beta_{v,1}\sigma_u^2\Sigma_0}{4\sigma_u^2\sigma_\varepsilon^2+3\beta_{v,1}^2\Sigma_0\sigma_u^2+4\beta_{v,1}^2\Sigma_0\sigma_\varepsilon^2}. \tag{B10}$$

By (B9),

$$\lambda_2 = \sigma_u^{-1}\sqrt{\left(\frac{1}{2}+\frac{\lambda_2\delta_s}{2}\beta_{v,1}\right)\frac{1}{\Sigma_0^{-1}+\beta_{v,1}^2\sigma_u^{-2}}-\left(\frac{1}{2}+\frac{\lambda_2\delta_s}{2}\beta_{v,1}\right)^2\frac{1}{\Sigma_0^{-1}+\beta_{v,1}^2\sigma_u^{-2}}-\left(\lambda_2\delta_s\right)^2\sigma_\varepsilon^2}.$$

Inserting (B10) into the above expression, we obtain

$$\lambda_2^2 = \Sigma_0\frac{\left(2\sigma_u^4+4\sigma_\varepsilon^4+5\sigma_u^2\sigma_\varepsilon^2\right)\Sigma_0^2\beta_{v,1}^4+\left(8\sigma_u^2\sigma_\varepsilon^4+5\sigma_u^4\sigma_\varepsilon^2\right)\Sigma_0\beta_{v,1}^2+4\sigma_u^4\sigma_\varepsilon^4}{\left(\beta_{v,1}^2\Sigma_0+\sigma_u^2\right)\left(3\sigma_u^2\Sigma_0\beta_{v,1}^2+4\sigma_\varepsilon^2\Sigma_0\beta_{v,1}^2+4\sigma_u^2\sigma_\varepsilon^2\right)^2}, \tag{B11}$$

which gives the expression of λ_2 in Proposition 2.

We can rewrite equation (24) as

$$2\lambda_2\left(2\beta_{v,1}\lambda_1-1\right)=\left[\beta_{v,1}\left(\frac{2}{3}\lambda_1+\lambda_2\delta_s\left(1+\frac{4\sigma_\varepsilon^2}{3\sigma_u^2}\right)\right)-1\right]\times\left[\frac{2}{3}\lambda_1+\lambda_2\delta_s\left(1+\frac{4\sigma_\varepsilon^2}{3\sigma_u^2}\right)\right]. \tag{B12}$$

We then want to take square on both sides of (B12) in order to use (B11) to substitute λ_2^2. Doing this requires that the terms $2\beta_{v,1}\lambda_1-1$ and $\beta_{v,1}\left(\frac{2}{3}\lambda_1+\lambda_2\delta_s\left(1+\frac{4\sigma_\varepsilon^2}{3\sigma_u^2}\right)\right)-1$ have the same sign, that is,

$$\left(2\beta_{v,1}\lambda_1-1\right)\left[\beta_{v,1}\left(\frac{2}{3}\lambda_1+\lambda_2\delta_s\left(1+\frac{4\sigma_\varepsilon^2}{3\sigma_u^2}\right)\right)-1\right]\geq 0.$$

Inserting the expression of λ_1 and $\lambda_2\delta_s$ in (B8) and (B10) into the above condition, we find that the above inequality is equivalent to requiring

$$\beta_{v,1}\leq\frac{\sigma_u}{\sqrt{\Sigma_0}}.$$

Thus, given $\beta_{v,1}\leq\frac{\sigma_u}{\sqrt{\Sigma_0}}$, we can take square of (B12), and set

$$4\lambda_2^2\left(2\beta_{v,1}\lambda_1-1\right)^2-\left[\beta_{v,1}\left(\frac{2}{3}\lambda_1+\lambda_2\delta_s\left(1+\frac{4\sigma_\varepsilon^2}{3\sigma_u^2}\right)\right)-1\right]^2\times\left[\frac{2}{3}\lambda_1+\lambda_2\delta_s\left(1+\frac{4\sigma_\varepsilon^2}{3\sigma_u^2}\right)\right]^2=0.$$

Inserting the expression of λ_1, $\lambda_2\delta_s$ and λ_2^2 in (B8), (B10), and (B11) into the above equation, we have the 7^{th} order polynomial of $\beta_{v,1}^2$ as follows:

$$f\left(\beta_{v,1}^2\right)=A_7\beta_{v,1}^{14}+A_6\beta_{v,1}^{12}+A_5\beta_{v,1}^{10}+A_4\beta_{v,1}^{8}+A_3\beta_{v,1}^{6}+A_2\beta_{v,1}^{4}+A_1\beta_{v,1}^{2}+A_0=0, \tag{B13}$$

where

$$\begin{aligned}
A_7 &= \Sigma_0^7\left(2\sigma_u^4+4\sigma_\varepsilon^4+5\sigma_u^2\sigma_\varepsilon^2\right)\left(3\sigma_u^2+4\sigma_\varepsilon^2\right)^2, && \text{(B14)}\\
A_6 &= 2\Sigma_0^6\sigma_u^2\left(4\sigma_\varepsilon^2-\sigma_u^2\right)\left(3\sigma_u^2+4\sigma_\varepsilon^2\right)\left(3\sigma_u^4+6\sigma_\varepsilon^4+8\sigma_u^2\sigma_\varepsilon^2\right), && \text{(B15)}\\
A_5 &= -\Sigma_0^5\sigma_u^6\left(27\sigma_u^6+336\sigma_\varepsilon^6+524\sigma_u^2\sigma_\varepsilon^4+246\sigma_u^4\sigma_\varepsilon^2\right), && \text{(B16)}\\
A_4 &= 4\Sigma_0^4\sigma_u^6\left(3\sigma_u^8-144\sigma_\varepsilon^8-304\sigma_u^2\sigma_\varepsilon^6-182\sigma_u^4\sigma_\varepsilon^4-23\sigma_u^6\sigma_\varepsilon^2\right), && \text{(B17)}\\
A_3 &= -\Sigma_0^3\sigma_u^8\left(\sigma_u^8+704\sigma_\varepsilon^8+752\sigma_u^2\sigma_\varepsilon^6+76\sigma_u^4\sigma_\varepsilon^4-57\sigma_u^6\sigma_\varepsilon^2\right), && \text{(B18)}\\
A_2 &= -4\Sigma_0^2\sigma_u^{10}\sigma_\varepsilon^2\left(\sigma_u^6+48\sigma_\varepsilon^6-24\sigma_u^2\sigma_\varepsilon^4-31\sigma_u^4\sigma_\varepsilon^2\right), && \text{(B19)}\\
A_1 &= -4\Sigma_0\sigma_u^{12}\sigma_\varepsilon^4\left(\sigma_u^4-32\sigma_\varepsilon^4-36\sigma_u^2\sigma_\varepsilon^2\right), && \text{(B20)}\\
A_0 &= 64\sigma_u^{14}\sigma_\varepsilon^8. && \text{(B21)}
\end{aligned}$$

The final requirement is to ensure that a root to the polynomial also satisfies the two SOC's, (13) and (25). Given the expression of λ_2 in Proposition 2, (13) is redundant. Also, (25) implies $\beta_{v,1}>0$, because (25) implies $\lambda_1>0$, which by (B8), in turn implies $\beta_{v,1}>0$. So, the final constraint on $\beta_{v,1}$ is $0<\beta_{v,1}\leq\frac{\sigma_u}{\sqrt{\Sigma_0}}$ and condition (25).

B.4 Proof of Proposition 3

When σ_ε is small: By Proposition 1, when σ_ε is small, there is a mixed strategy equilibrium. The task is to show that there is no pure strategy equilibrium. By (24) and the fact $\beta_{v,1} > 0$ in a pure strategy equilibrium, we have

$$\beta_{v,1} = \frac{1 - \frac{\lambda_1 + \lambda_2\delta_s - \lambda_2\delta_{y_1}}{2\lambda_2}}{2\left[\lambda_1 - \frac{\left(\lambda_1 + \lambda_2\delta_s - \lambda_2\delta_{y_1}\right)^2}{4\lambda_2}\right]} > 0. \tag{B22}$$

Note that the denominator is the SOC in (25), which is positive. So, we must have

$$1 - \frac{\lambda_1 + \lambda_2\delta_s - \lambda_2\delta_{y_1}}{2\lambda_2} > 0 \Rightarrow 4\lambda_2^2 - \left(\lambda_1 + \lambda_2\delta_s - \lambda_2\delta_{y_1}\right)^2 > 0.$$

Using (19) we can rewrite the above inequality as follows:

$$4\lambda_2^2 - \left(\frac{2}{3}\lambda_1 + \lambda_2\delta_s\left(1 + \frac{4\sigma_\varepsilon^2}{3\sigma_u^2}\right)\right)^2 > 0. \tag{B23}$$

Plugging the expression of λ_1, $\lambda_2\delta_s$, and λ_2^2 in (B8), (B10), and (B11) into the left-hand-side (LHS) of (B23), we find that (B23) is equivalent to

$$\begin{aligned}&\left(16\beta_{v,1}^4\Sigma_0^2 + 32\beta_{v,1}^2\Sigma_0\sigma_u^2 + 16\sigma_u^4\right)\sigma_\varepsilon^4 + \left(8\beta_{v,1}^4\Sigma_0^2\sigma_u^2 - 4\beta_{v,1}^6\Sigma_0^3 + 12\beta_{v,1}^2\Sigma_0\sigma_u^4\right)\sigma_\varepsilon^2\\ &-\beta_{v,1}^2\Sigma_0\sigma_u^2\left(\beta_{v,1}^2\Sigma_0 - \sigma_u^2\right)^2 > 0.\end{aligned} \tag{B24}$$

We prove that the above condition is not satisfied in a pure strategy equilibrium, as $\sigma_\varepsilon \to 0$. Proposition 2 implies that in a pure strategy equilibrium, $\beta_{v,1}^2 \in \left(0, \frac{\sigma_u^2}{\Sigma_0}\right]$. So, as $\sigma_\varepsilon \to 0$, the first two terms of the LHS of (B24) go to 0. Thus, if as $\sigma_\varepsilon \to 0$, $\beta_{v,1}^2$ does not go to 0 or $\frac{\sigma_u^2}{\Sigma_0}$ in a pure strategy equilibrium, then the third term of the LHS of (B24) is strictly negative, which proves our statement.

Now we consider the two cases that $\beta_{v,1}^2$ converges to 0 or to $\frac{\sigma_u^2}{\Sigma_0}$ as $\sigma_\varepsilon \to 0$. We will show that both lead to contradictions to a pure strategy equilibrium.

Note that if $\sigma_\varepsilon = 0$, the polynomial (B13) is negative at $\frac{\sigma_u^2}{\Sigma_0}$; that is, $f\left(\frac{\sigma_u^2}{\Sigma_0}\right) = -16\sigma_u^{22} < 0$ if $\sigma_\varepsilon = 0$. Thus, if for any sequence of $\sigma_\varepsilon \to 0$, we have $\beta_{v,1}^2 \to \frac{\sigma_u^2}{\Sigma_0}$ in a pure strategy equilibrium, then we must have $f\left(\beta_{v,1}^2\right) \to -16\sigma_u^{22} < 0$, which contradicts with Proposition 2 which says that $f\left(\beta_{v,1}^2\right) \equiv 0$ in a pure strategy equilibrium. Thus, $\beta_{v,1}^2 \not\to \frac{\sigma_u^2}{\Sigma_0}$ as $\sigma_\varepsilon \to 0$.

Suppose $\beta_{v,1}^2 \to 0$ in a pure strategy equilibrium for some sequence of $\sigma_\varepsilon^2 \to 0$. By (24), we have

$$\left(\frac{2}{3}\lambda_1 + \lambda_2\delta_s\left(1 + \frac{4\sigma_\varepsilon^2}{3\sigma_u^2}\right)\right)^2 > \left(1 - 2\frac{\beta_1\Sigma_0}{\beta_1^2\Sigma_0 + \sigma_u^2}\beta_1\right)^2 4\lambda_2^2.$$

Combining the above condition with condition (B23), we know $\left(\frac{2}{3}\lambda_1+\lambda_2\delta_s\left(1+\frac{4\sigma_\varepsilon^2}{3\sigma_u^2}\right)\right)^2$ has the same order as λ_2^2:

$$O\left(\left(\frac{2}{3}\lambda_1+\lambda_2\delta_s\left(1+\frac{4\sigma_\varepsilon^2}{3\sigma_u^2}\right)\right)^2\right)=O\left(\lambda_2^2\right).$$

Substituting into the above equation the expression of λ_1, $\lambda_2\delta_s$, and λ_2^2 from (B8), (B10), and (B11) and matching the highest-order terms, we can show that $\beta_{v,1}^2$ has the same order as σ_ε^4. As a result, by (B8), $\lambda_1\to 0$; by (B10), $\lambda_2\delta_s$ goes to a positive finite number; and by (B11), λ_2 goes to a positive finite number. This in turn implies the SOC (25) is violated. Specifically, by (19), the SOC is equivalent to

$$\lambda_1-\frac{\left(\frac{2}{3}\lambda_1+\lambda_2\delta_s\left(1+\frac{4\sigma_\varepsilon^2}{3\sigma_u^2}\right)\right)^2}{4\lambda_2}>0. \tag{B25}$$

However, as $\sigma_\varepsilon^2\to 0$, we have $\lambda_1-\frac{\left(\frac{2}{3}\lambda_1+\lambda_2\delta_s\left(1+\frac{4\sigma_\varepsilon^2}{3\sigma_u^2}\right)\right)^2}{4\lambda_2}\to-\frac{(\lambda_2\delta_s)^2}{4\lambda_2}<0$, a contradiction.

When σ_ε is large: By Proposition 1, when σ_ε is sufficiently large, there is no mixed strategy equilibrium. The task is to show that a linear pure strategy equilibrium exists and is unique.

By equations (B14)–(B21), we have $A_7>0$, $A_6>0$, $A_5<0$, $A_4<0$, $A_3<0$, $A_2<0$, $A_1>0$, and $A_0>0$, when σ_ε^2 is sufficiently large. Thus, by Descartes' Rule of Signs, there are at most two positive roots of $\beta_{v,1}^2$.

By equation (B13), we have

$$\begin{aligned}
f(0) &= 64\sigma_u^{14}\sigma_\varepsilon^8>0,\\
\lim_{\beta_{v,1}^2\to\infty} f\left(\beta_{v,1}^2\right) &\propto \Sigma_0^7\left(2\sigma_u^4+4\sigma_\varepsilon^4+5\sigma_u^2\sigma_\varepsilon^2\right)\left(3\sigma_u^2+4\sigma_\varepsilon^2\right)^2\times\infty>0.
\end{aligned}$$

In addition, as $\sigma_\varepsilon^2\to\infty$, $f\left(\frac{\sigma_u^2}{\Sigma_0}\right)\propto-1024\sigma_u^{14}\sigma_\varepsilon^8<0$. So, there is exactly one root of $\beta_{v,1}^2$ in the range of $\left(0,\frac{\sigma_u^2}{\Sigma_0}\right)$ and one root in the range of $\left(\frac{\sigma_u^2}{\Sigma_0},\infty\right)$. Given that in a pure strategy equilibrium, we require $0<\beta_{v,1}^2\leq\frac{\sigma_u}{\Sigma_0}$ by Proposition 2, only the small root is a possible equilibrium candidate (which is indeed an equilibrium if the SOC is also satisfied).

Finally, we can show that the small root of $\beta_{v,1}^2\in\left(0,\frac{\sigma_u^2}{\Sigma_0}\right)$ satisfies the SOC as $\sigma_\varepsilon^2\to\infty$. Specifically, by (B25), the SOC is

$$\lambda_1-\frac{\left(\frac{2}{3}\lambda_1+\lambda_2\delta_s\left(1+\frac{4\sigma_\varepsilon^2}{3\sigma_u^2}\right)\right)^2}{4\lambda_2}>0\Leftrightarrow$$

$$16\lambda_2^2\lambda_1^2 - \left(\frac{2}{3}\lambda_1 + \lambda_2\delta_s\left(1 + \frac{4\sigma_\varepsilon^2}{3\sigma_u^2}\right)\right)^4 > 0.$$

Plugging the expression of λ_1, $\lambda_2\delta_s$, and λ_2^2 in (B8), (B10) and (B11) into the LHS of the above condition, we can show that the above condition holds if and only if

$$B_4\sigma_\varepsilon^8 + B_3\sigma_\varepsilon^6 + B_2\sigma_\varepsilon^4 + B_1\sigma_\varepsilon^2 + B_0 > 0, \tag{B26}$$

where,

$$\begin{aligned}
B_4 &= 768\beta_{v,1}^{10}\Sigma_0^5 + 4096\beta_{v,1}^8\Sigma_0^4\sigma_u^2 + 8704\beta_{v,1}^6\Sigma_0^3\sigma_u^4 + 9216\beta_{v,1}^4\Sigma_0^2\sigma_u^6 + 4864\beta_{v,1}^2\Sigma_0\sigma_u^8 + 1024\sigma_u^{10},\\
B_3 &= 2048\beta_{v,1}^{10}\Sigma_0^5\sigma_u^2 + 8704\beta_{v,1}^8\Sigma_0^4\sigma_u^4 + 13824\beta_{v,1}^6\Sigma_0^3\sigma_u^6 + 9728\beta_{v,1}^4\Sigma_0^2\sigma_u^8 + 2560\beta_{v,1}^2\Sigma_0\sigma_u^{10},\\
B_2 &= 2144\beta_{v,1}^{10}\Sigma_0^5\sigma_u^4 + 6720\beta_{v,1}^8\Sigma_0^4\sigma_u^6 + 6912\beta_{v,1}^6\Sigma_0^3\sigma_u^8 + 2240\beta_{v,1}^4\Sigma_0^2\sigma_u^{10} - 96\beta_{v,1}^2\Sigma_0\sigma_u^{12},\\
B_1 &= 1056\beta_{v,1}^{10}\Sigma_0^5\sigma_u^6 + 2112\beta_{v,1}^8\Sigma_0^4\sigma_u^8 + 912\beta_{v,1}^6\Sigma_0^3\sigma_u^{10} - 160\beta_{v,1}^4\Sigma_0^2\sigma_u^{12} - 16\beta_{v,1}^2\Sigma_0\sigma_u^{14},\\
B_0 &= 207\beta_{v,1}^{10}\Sigma_0^5\sigma_u^8 + 180\beta_{v,1}^8\Sigma_0^4\sigma_u^{10} - 54\beta_{v,1}^6\Sigma_0^3\sigma_u^{12} - 12\beta_{v,1}^4\Sigma_0^2\sigma_u^{14} - \beta_{v,1}^2\Sigma_0\sigma_u^{16}.
\end{aligned}$$

Given that $\beta_{v,1}$ is bounded, we have that as σ_ε^2 is large, the LHS of condition (B26) is determined by $B_4\sigma_\varepsilon^8$, which is always positive: $B_4\sigma_\varepsilon^8 > 1024\sigma_u^{10}\sigma_\varepsilon^8 > 0$.

B.5 Proof of Corollary 1

Now suppose $\sigma_\varepsilon \to \infty$. By Proposition 3, as σ_ε is large, there is a unique linear equilibrium, which is a pure strategy equilibrium. In a pure strategy equilibrium, we always have $f\left(\beta_{v,1}^2\right) = 0$. If we rewrite the polynomial f as a polynomial in terms of σ_ε, we must have that as $\sigma_\varepsilon \to \infty$, the coefficients on the highest order of σ_ε goes to 0. This exercise yields the following condition that as $\sigma_\varepsilon \to \infty$, we have

$$64\Sigma_0^7\beta_{v,1}^{14} + 192\Sigma_0^6\sigma_u^2\beta_{v,1}^{12} - 576\Sigma_0^4\sigma_u^6\beta_{v,1}^8 - 704\Sigma_0^3\sigma_u^8\beta_{v,1}^6 - 192\Sigma_0^2\sigma_u^{10}\beta_{v,1}^4 + 128\Sigma_0\sigma_u^{12}\beta_{v,1}^2 + 64\sigma_u^{14} \to 0. \tag{B27}$$

Define $x \equiv \beta_{v,1}^2\frac{\Sigma_0}{\sigma_u^2} \in [0,1]$ in a pure strategy equilibrium. Condition (B27) becomes

$$-2x - x^2 + x^3 + 1 \to 0, \text{ as } \sigma_\varepsilon \to \infty. \tag{B28}$$

That is, as $\sigma_\varepsilon \to \infty$, we must have that (B28) holds.

In a standard Kyle setting, the unique equilibrium is defined by

$$-2x^* - x^{*2} + x^{*3} + 1 = 0. \tag{B29}$$

Specifically, Proposition 1 of Huddart, Hughes, and Levine (2001) characterizes the equilibrium in a two-period Kyle model by a cubic in terms of K,

$$8K^3 - 4K^2 - 4K + 1 = 0. \tag{B30}$$

By the expressions of $\beta_1 = \frac{2K-1}{4K-1}\frac{1}{\lambda_1}$ and $\lambda_1 = \frac{\sqrt{2K(2K-1)}}{4K-1}\frac{\sqrt{\Sigma_0}}{\sigma_u}$ in Proposition 1 of Huddart, Hughes, and Levine (2001), we have $K = \frac{1}{2(1-x^*)}$, where $x^* = \beta_1^2 \frac{\Sigma_0}{\sigma_u^2}$. Then, equation (B30) is equivalent to equation (B29). Given that $-2x - x^2 + x^3 + 1$ is monotone and continuous in the range of $[0,1]$, we have $x \to x^*$ as $\sigma_\varepsilon \to \infty$, by conditions (B28) and (B29).

B.6 Proof of Proposition 4

We here give the expression of the variables in the proposition. The comparison follows from setting $\sigma_\varepsilon = 0$ and $\sigma_\varepsilon = \infty$ in these expressions and from straightforward computations.

Setting $\sigma_\varepsilon = 0$ in Proposition 1 yields $\sigma_z = \sqrt{\frac{1}{39}}\sigma_u$ and $\beta_{v,1} = \frac{2}{\sqrt{13}}\frac{\sigma_u}{\sqrt{\Sigma_0}}$. Plugging these two expressions of σ_z and $\beta_{v,1}$ into the expressions of λ_1 and λ_2 in Proposition 1 gives λ_1^0 and λ_2^0. In a pure strategy equilibrium, λ_1 and λ_2 are given by Proposition 2. Setting $\sigma_\varepsilon = \infty$ yields the expressions of λ_1^{Kyle} and λ_2^{Kyle}.

Direct computation shows that in a mixed strategy equilibrium, the price discovery variables are given by

$$\begin{aligned}
\Sigma_1^{mixed} &= \frac{\left(\sigma_z^2 + \sigma_u^2\right)\Sigma_0}{\beta_{v,1}^2 \Sigma_0 + \sigma_z^2 + \sigma_u^2}, \\
\Sigma_2^{mixed} &= \frac{\left(4\sigma_u^4 + 4\sigma_u^2\sigma_z^2 + \sigma_u^2\delta_s^2\sigma_z^2 + 4\sigma_u^2\delta_s^2\sigma_\varepsilon^2 + 4\delta_s^2\sigma_z^2\sigma_\varepsilon^2\right)\lambda_2^2\Sigma_0}{\left(\begin{array}{c} \left(\Sigma_0\lambda_2^2\sigma_u^2\delta_s^2 + 4\Sigma_0\lambda_2^2\sigma_u^2 + 4\Sigma_0\lambda_2^2\delta_s^2\sigma_\varepsilon^2\right)\beta_{v,1}^2 + 2\lambda_2\Sigma_0\sigma_u^2\delta_s\beta_{v,1} \\ +\lambda_2^2\left(4\sigma_u^4 + 4\sigma_u^2\sigma_z^2 + \sigma_u^2\delta_s^2\sigma_z^2 + 4\sigma_u^2\delta_s^2\sigma_\varepsilon^2 + 4\delta_s^2\sigma_z^2\sigma_\varepsilon^2\right) + \Sigma_0\left(\sigma_u^2 + \sigma_z^2\right) \end{array}\right)}.
\end{aligned}$$

Plugging $\sigma_\varepsilon = 0, \sigma_z = \sqrt{\frac{1}{39}}\sigma_u$, and $\beta_{v,1} = \frac{2}{\sqrt{13}}\frac{\sigma_u}{\sqrt{\Sigma_0}}$ into the above expressions yields Σ_1^0 and Σ_2^0. In a pure strategy equilibrium, we can compute

$$\begin{aligned}
\Sigma_1^{pure} &= \frac{\sigma_u^2\Sigma_0}{\sigma_u^2 + \beta_{v,1}^2\Sigma_0}, \\
\Sigma_2^{pure} &= \frac{1}{\Sigma_0^{-1} + \beta_{v,1}^2\sigma_u^{-2} + \left(\frac{1}{2\lambda_2} + \frac{\delta_s}{2}\beta_{v,1}\right)^2\left(\delta_s^2\sigma_\varepsilon^2 + \sigma_u^2\right)^{-1}}.
\end{aligned}$$

Setting $\sigma_\varepsilon = \infty$ in Proposition 2, computing $\beta_{v,1}$, λ_2 and δ_s, and inserting these expressions into the above expressions, we have the expressions of Σ_1^{Kyle} and Σ_2^{Kyle}.

Finally, we present the profit expressions. By (15), the fundamental investor's *ex-ante* expected period-2 profit is

$$E\left(\pi_{F,2}\right) = \frac{E\left[v - p_1 - \lambda_2\left(-\delta_y y_1 + \delta_s x_1\right)\right]^2}{4\lambda_2}.$$

Using equations (5) and (8), we can show

$$E\left(\pi_{F,2}\right)=\frac{\left[1-\left(\lambda_1-\lambda_2\delta_y+\lambda_2\delta_s\right)\beta_{v,1}\right]^2\Sigma_0+\left(\lambda_1-\lambda_2\delta_y\right)^2\sigma_u^2+\left(\lambda_1-\lambda_2\delta_y+\lambda_2\delta_s\right)^2\sigma_z^2}{4\lambda_2}.$$

For $E\left(\pi_{F,2}^0\right)$, we set $\sigma_\varepsilon=0$, compute $\sigma_z,\beta_{v,1},\lambda_1,\lambda_2,\delta_s$, and δ_y in Proposition 1, and insert these expressions into the above equation. For $E\left(\pi_{F,2}^{Kyle}\right)$, we set $\sigma_\varepsilon=\infty$ in Proposition 2 and compute the relevant parameters accordingly.

To give the expression of $E\left(\pi_{F,1}\right)$, we first compute $E\left(\pi_{F,1}+\pi_{F,2}\right)$, and then use the above computed $E\left(\pi_{F,2}\right)$ to compute $E\left(\pi_{F,1}\right)=E\left(\pi_{F,1}+\pi_{F,2}\right)-E\left(\pi_{F,2}\right)$. Using equation (22), we can show that in a mixed strategy equilibrium,

$$E\left(\pi_{F,1}^{mixed}+\pi_{F,2}^{mixed}\right)=\frac{\Sigma_0+\sigma_u^2\lambda^2\left(1-\delta_y\right)^2}{4\lambda}$$

while in a pure strategy equilibrium

$$E\left(\pi_{F,1}^{pure}+\pi_{F,2}^{pure}\right)=\frac{\left(1-\frac{\lambda_1+\lambda_2\delta_s-\lambda_2\delta_y}{2\lambda_2}\right)^2}{4\left(\lambda_1-\frac{\left(\lambda_1+\lambda_2\delta_s-\lambda_2\delta_y\right)^2}{4\lambda_2}\right)}\Sigma_0+\frac{\Sigma_0+\sigma_u^2\left(\lambda_1-\lambda_2\delta_y\right)^2}{4\lambda_2}.$$

Then, setting $\sigma_\varepsilon=0$ and $\sigma_\varepsilon=\infty$ in the above two expressions gives $E\left(\pi_{F,1}^0+\pi_{F,2}^0\right)$ and $E\left(\pi_{F,1}^{Kyle}+\pi_{F,2}^{Kyle}\right)$, respectively.

References

Admati, A. and P. Pfleiderer (1989): "Divide and conquer: A theory of intraday and day-of-the-week mean effects." *Review of Financial Studies*, 2, 189–223.

Attari, M., A. S. Mello, and M. E. Ruckes (2005): "Arbitraging Arbitrageurs," *Journal of Finance*, 60, 2471–2511.

Back, K. and S. Baruch (2004): "Information in Securities Markets: Kyle Meets Glosten and Milgrom," *Econometrica*, 72, 433–465.

Baruch, S. and L. R. Glosten (2013): "Fleeting Orders," Working paper.

Bernhardt, D. and B. Taub (2008): "Front-Running Dynamics," *Journal of Economic Theory*, 138, 288–296.

Bhattacharya, U., C. W. Holden, and S. Jacobsen (2012): "Penny Wise, Dollar Foolish: Buy-Sell Imbalances On and Around Round Numbers," *Management Science*, 58, 413–431.

Biais, B., T. Foucault, and S. Moinas (2015): "Equilibrium Fast Trading," *Journal of Financial Economics*, 116, 292–313.

Bond, P., A. Edmans, and I. Goldstein (2012): "The Real Effects of Financial Markets," *Annual Reviews of Financial Economics*, 4, 339–360.

Bond, P., I. Goldstein, and E. S. Prescott (2010): "Market-Based Corrective Actions," *Review of Financial Studies*, 23, 781–820.

Brogaard, J., T. Hendershott, and R. Riordan (2014): "High-Frequency Trading and Price Discovery," *Review of Financial Studies*, 27, 2267–2306.

Brunnermeier, M. K. (2005): "Information Leakage and Market Efficiency," *Review of Financial Studies*, 18, 417–457.

Brunnermeier, M. K. and L. H. Pedersen (2005): "Predatory Trading," *Journal of Finance*, 60, 1825–1863.

Budish, E., P. Cramton, and J. Shim (2015): "The High-Frequency Trading Arms Race: Frequent Batch Auctions as a Market Design Response," *Quarterly Journal of Economics (Forthcoming)*.

Buffa, A. (2013): "Insider Trade Disclosure, Market Efficiency, and Liquidity," Working paper.

Cao, H. H., M. D. Evans, and R. K. Lyons (2006): "Inventory Information," *Journal of Business*, 79, 325–364.

Carlin, B. I., M. S. Lobo, and S. Viswanathan (2007): "Episodic Liquidity Crisis: Cooperative and Predatory Trading," *Journal of Finance*, 62, 2235–2274.

Easley, D., M. M. L. de Prado, and M. O'Hara (2012): "The Volume Clock: Insights into the High-Frequency Paradigm," *Journal of Portfolio Management*, 39, 19–29.

Engelberg, J. E., A. V. Reed, and M. C. Ringgenberg (2012): "How are Shorts Informed? Short Sellers, News, and Information Processing," *Journal of Financial Economics*, 105, 260–278.

Foucault, T., J. Hombert, and I. Rosu (2015): "News Trading and Speed," *Journal of Finance (Forthcoming).*

Glosten, L. R. and P. R. Milgrom (1985): "Bid, Ask and Transaction Prices in a Specialist Market with Heterogeneously Informed Traders," *Journal of Financial Economics*, 14, 71–100.

Harris, L. (2003): *Trading and Exchanges: Market Microstructure for Practitioners*, Oxford University Press.

——— (2013): "What to Do about High-Frequency Trading," *Financial Analyst Journal*, 69, 6–9.

Hirschey, N. (2013): "Do High-Frequency Traders Anticipate Buying and Selling Pressure?" Working paper.

Hoffmann, P. (2014): "A dynamic limit order market with fast and slow traders," *Journal of Financial Economics*, 113, 156–169.

Huddart, S., J. S. Hughes, and C. B. Levine (2001): "Public Disclosure and Dissimulation of Insider Trades," *Econometrica*, 69, 665–681.

Jones, C. (2013): "What do we know about high-frequency trading?" Working paper.

Jovanovic, B. and A. J. Menkveld (2012): "Middlemen in Limit-Order Markets," Working paper.

Kim, O. and R. E. Verrecchia (1994): "Market Liquidity and Volume around Earnings Announcements," *Journal of Accounting and Economics*, 17, 41–67.

Korajczyk, R. and D. Murphy (2014): "High Frequency Market Making to Large Institutional Trades," Working paper.

Kyle, A. S. (1985): "Continuous Auctions and Insider Trading," *Econometrica*, 53, 1315–1335.

Lewis, M. (2014): *Flash Boys*, Norton.

Li, W. (2014): "High Frequency Trading with Speed Hierarchies," Working paper.

Madrigal, V. (1996): "Non-Fundamental Speculation," *Journal of Finance*, 51, 553–578.

O'Hara, M. (2003): "Presidential Address: Liquidity and Price Discovery," *Journal of Finance*, 58, 1335–1354.

Securities and Exchange Commission (2010): "Concept Release on Equity Market Structure," Technical report.

Stiglitz, J. (2014): "Tapping the Brakes: Are Less Active Markets Safer and Better for the Economy?" Speech at the 2014 Financial Markets Conference at the Federal Reserve Bank of Atlanta.

Tong, L. (2015): "A Blessing or a Curse? The Impact of High Frequency Trading on Institutional Investors," Working paper.

van Kervel, V. and A. Menkveld (2015): "High-Frequency Trading around Large Institutional Orders," Working paper.

Verrecchia, R. E. (1982): "Information Acquisition in a Noisy Rational Expectations Economy," *Econometrica*, 50, 1415–1430.

Vives, X. (2008): *Information and Learning in Markets: The Impact of Market Microstructure*, Princeton University Press, Princeton and Oxford.

Weller, B. (2015): "Efficient Prices at Any Cost: Does Algorithmic Trading Deter Information Acquisition?" Working paper.

Yueshen, B. Z. (2015): "Uncertain Market Making," Working paper.